

Limit theorems for random walks in dynamic random environment

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We study a general class of random walks driven by a uniquely ergodic Markovian environment. Under a coupling condition on the environment we obtain strong ergodicity properties and concentration inequalities for the environment as seen from the position of the walker, i.e the environment process. We also obtain ergodicity of the uniquely ergodic measure of the environment process as well as continuity as a function of the jump rates of the walker.

As a consequence we obtain several limit theorems, such as law of large numbers, Einstein relation, central limit theorem and concentration properties for the position of the walker.

Keywords: environment process, coupling, random walk, concentration estimates, backwards martingales.

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1 Introduction

In recent days random walks in dynamic random environment have been studied by several authors. Motivation comes among others from non-equilibrium statistical mechanics -derivation of Fourier law- [4] and large deviation theory [9]. In principle random walk in dynamic random environment contains as a particular case random walk in static random environment. However, mostly, in turning to dynamic environments, authors concentrate more on environments with sufficient mixing properties. In that case the fact that the environment is dynamic helps to obtain self-averaging properties that ensure standard limiting behavior of the walk, i.e., law of large numbers and central limit theorem.

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In the study of the limiting behavior of the walker, the environment process, i.e., the environment as seen from the position of the walker plays a crucial role. See also [6], [8] for the use of the environment process in related context. In a translation invariant setting the environment process is a Markov process and its ergodic properties fully determine corresponding ergodic properties of the walk, since the position of the walker equals an additive function of the environment process plus a controllable martingale.

The main theme of this paper is precisely the study of the following natural question: if the environment is uniquely ergodic, with a sufficient speed of mixing, then the environment process shares similar properties. In several works ([2], [3], [1]) this transfer of “good properties of the environment” to “similar properties of the environment process” is made via a perturbative argument, and therefore holds only in a regime where the environment and the walker are weakly coupled. An exception is [5] in the case of an environment consisting of independent Markov processes.

In this paper we consider the context of general Markovian uniquely ergodic environments, which are such that the semigroup contracts at a minimal speed in norm of variation type. Examples of such environments include interacting particle systems in “the $M < \epsilon$ regime” [7] and weakly interacting diffusion processes on a compact manifold. Our conditions on the environment are formulated in the language of coupling. More precisely, we impose that for the environment there exists a coupling such that the distance between every pair of initial configurations in this coupling decays fast enough so that multiplied with t^d it is still integrable in time. As a result we then obtain that for the environment process there exists a coupling such that the distance between every pair of initial configurations in this coupling decays at a speed which is at least integrable in time. In fact we show more, namely in going from the environment to the environment process we essentially loose a factor t^d . E.g., if for the environment there is a coupling where the distance decays exponentially, then the same holds for the environment process (with possibly another rate).

Once we have controllable coupling properties of the environment process, we can draw strong conclusions for the position of the walker. More precisely, we prove a law of large numbers with an asymptotic speed that depends continuously on the rates, and a central limit theorem with a controllable asymptotic covariance matrix. We also prove recurrence in $d = 1$ under condition of zero speed and transience when the speed is non zero, as well as an Einstein relation.

Along the way, we develop a general formalism to derive concentration inequalities for functions of a Markov process, based on martingales adapted to quantities like the position of the walker, which is not exactly an additive functional of the environment process. Concentration inequalities for the position of the walker are crucial to obtain transience or recurrence.

The formalism to derive these concentration inequalities is in the spirit of [10] but based now on “backwards” martingales. These “backwards” martingales are different from the classical martingales associated to the generator, or arising in the “martingale problem”. Using them leads however to a controllable expression of the variance of the walker, as well as for additive functionals of the environment process. We believe that the use of these backwards martingales are interesting per se, and can also play an important role in controlling the deviation from macroscopic behavior (such as

hydrodynamic limits) in the context of interacting particle systems.

Our paper is organized as follows. The model and necessary notation are introduced in Section 2. Section 3 is dedicated to lift properties of the environment to the environment process. Especially Theorem 3.1 is of great importance and is used frequently in the later sections. Based on the results in Section 3 the Law of Large Numbers and an Einstein Relation are obtained in Section 4. A functional Central Limit Theorem is proven in Section 5, utilizing the martingale approach detailed in the Appendix. Section 6 further applies the general methods from the Appendix to the specific case of a random walk in dynamic random environment to obtain concentration estimates and therewith statements about recurrence and transience. Finally the Appendix is dedicated to the study of the non-time-homogeneous “backwards” martingales and the concentration inequalities they facilitate.

2 The model

2.1 Environment

We are interested in studying a random walk $(X_t)_{t \geq 0}$ on the lattice \mathbb{Z}^d which is driven by a second processes $(\eta_t)_{t \geq 0}$ on $E^{\mathbb{Z}^d}$, the (dynamic) environment. This can be interpreted as the random walk moving through the environment, and its transition rates being determined by the local environment around the random walk.

To become more precise, the *environment* $(\eta_t)_{t \geq 0}$ is a Feller Process on the state space $\Omega := E^{\mathbb{Z}^d}$, where (E, ρ) is a compact Polish space with metric ρ (examples in mind are $E = \{0, 1\}$ or $E = [0, 1]$). We assume that the distance ρ on E is bounded from above by 1. The generator of the Markov process is denoted by L^E and its semigroup by S_t^E , both considered on the space of continuous functions $\mathcal{C}(\Omega; \mathbb{R})$. We assume that the environment is translation invariant, i.e.

$$\mathbb{P}_\eta^E(\theta_x \eta_t \in \cdot) = \mathbb{P}_{\theta_x \eta}^E(\eta_t \in \cdot)$$

with θ_x denoting the shift operator $\theta_x \eta(y) = \eta(y - x)$ and \mathbb{P}_η^E the path space measure of the process $(\eta_t)_{t \geq 0}$ starting from η . Later on we will formulate precise conditions necessary to obtain our results.

2.2 Lipschitz functions

For practical purposes, we introduce the set of pairs in Ω which differ at one specific site:

$$(\Omega \times \Omega)_x := \{(\eta, \xi) \in \Omega^2 : \eta(x) \neq \xi(x) \text{ and } \eta(y) = \xi(y) \ \forall y \in \mathbb{Z}^d \setminus \{x\}\}, \quad x \in \mathbb{Z}^d.$$

Definition 2.1. For any $f : \Omega \rightarrow \mathbb{R}$, we denote by $\delta_f(x)$ the Lipschitz-constant of f when only site x is changed with respect to the distance ρ , i.e.

$$\delta_f(x) := \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \frac{f(\eta) - f(\xi)}{\rho(\eta(x), \xi(x))}.$$

We write

$$\|f\| := \sum_{x \in \mathbb{Z}^d} \delta_f(x). \quad (1)$$

Note that $\|f\| < \infty$ implies that f is globally bounded and that the value of f is uniformly weakly dependent on sites far away. A rougher semi-norm we also use is the oscillation (semi)-norm

$$\|f\|_{osc} := \sup_{\eta, \xi \in \Omega} (f(\eta) - f(\xi)).$$

Generally, those two semi-norms are related by the inequality $\|f\|_{osc} \leq \|f\|$.

2.3 The random walker and assumption on rates

The random walker X_t is a process on \mathbb{Z}^d , whose transition rates depend on the state of the environment as seen from the walker. More precisely, the rate to jump from site x to site $x + z$ given that the environment is in state η is $\alpha(\theta_{-x}\eta, z)$. We make two assumptions on the jump rates α . First, we guarantee that the walker X_t has first moments by assuming

$$\|\alpha\|_1 := \sum_{z \in \mathbb{Z}^d} \|z\| \sup_{\eta \in \Omega} |\alpha(\eta, z)| < \infty. \quad (2)$$

More generally, as sometimes higher moments are necessary, we write

$$\|\alpha\|_p^p := \sum_{z \in \mathbb{Z}^d} \|z\|^p \sup_{\eta \in \Omega} |\alpha(\eta, z)|, \quad p \geq 1.$$

Second, we limit the sensitivity of the rates to small changes in the environment by assuming that

$$\|\alpha\| := \sum_{z \in \mathbb{Z}^d} \|\alpha(\cdot, z)\| < \infty. \quad (3)$$

In Section 6, we impose the following stronger condition $\|\alpha\|_1 < \infty$:

$$\|\alpha\|_1 := \sum_{z \in \mathbb{Z}^d} \|z\| \|\alpha(\cdot, z)\| < \infty. \quad (4)$$

2.4 Environment process

While the random walker X_t itself is not a Markov process due to the dependence on the environment, the pair (η_t, X_t) is a Markov process with generator

$$Lf(\eta, x) = L^E f(\cdot, x)(\eta) + \sum_{z \in \mathbb{Z}^d} \alpha(\theta_{-x}\eta, z) [f(\eta, x + z) - f(\eta, x)],$$

corresponding semigroup S_t (considered on the space of functions continuous in $\eta \in \Omega$ and Lipschitz continuous in $x \in \mathbb{Z}^d$) and path space measure $\mathbb{P}_{\eta,x}$.

The environment as seen from the walker is of crucial importance to understand the asymptotic behaviour of the walker itself. This process, $(\theta_{-X_t}\eta_t)_{t \geq 0}$, is also called the environment process (this name is common in the literature, however in the context of this paper that name can easily be confused with the environment η_t), which is also a Markov process with generator

$$L^{EP}f(\eta) = L^E f(\eta) + \sum_{z \in \mathbb{Z}^d} \alpha(\eta, z) [f(\theta_{-z}\eta) - f(\eta)],$$

corresponding semigroup S_t^{EP} (on $\mathcal{C}(\Omega)$) and path space measure \mathbb{P}_η^{EP} . Notice that this process is well-defined only in the translation invariant context.

3 Ergodicity of the environment process

3.1 Assumptions on the environment

In this section we will show how we can use a given Markovian coupling of the environment with a fast enough coupling speed to obtain a strong ergodicity properties about the environment process.

Assumption 1a. *There exists a strong Markovian coupling $\hat{\mathbb{P}}^E$ of the environment (with corresponding expectation $\hat{\mathbb{E}}^E$) which satisfies*

$$\int_0^\infty t^d \sup_{\eta, \xi \in \Omega} \hat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)) dt < \infty.$$

This assumption is already sufficient to obtain the law of large numbers for the position of the walker and unique ergodicity of the environment process, but it does not give quite enough control on local fluctuations. The following stronger assumption remedies that.

Assumption 1b. *There exists a strong Markovian coupling $\hat{\mathbb{P}}^E$ of the environment (with corresponding expectation $\hat{\mathbb{E}}^E$) which satisfies*

$$\int_0^\infty t^d \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_0} \frac{\hat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(x), \eta_t^2(x))}{\rho(\eta(0), \xi(0))} dt < \infty.$$

Remark The following *telescoping* argument which will be used several times later on shows how Assumption 1b implies 1a:

Fix $\eta, \xi \in \Omega$. Let $(\zeta_n)_{n \geq 0}$ be a sequence in Ω with $\zeta_0 = \eta$, $\lim_{n \rightarrow \infty} \zeta_n = \xi$ and $|\{x \in \mathbb{Z}^d : \zeta_n(x) \neq \zeta_{n-1}(x)\}| \leq 1$ for all $n \geq 1$. This sequence interpolates between the configurations η and ξ by single site changes. Using such a sequence, we can *telescope over single site changes*, and using the triangle inequality for each n :

$$\hat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)) \leq \sum_{n \in \mathbb{N}} \hat{\mathbb{E}}_{\zeta_{n+1}, \zeta_n}^E \rho(\eta_t^1(0), \eta_t^2(0)).$$

We can assume that each site $x \in \mathbb{Z}^d$ is at most once changed in the sequence ζ_n . Therefore,

$$\sum_{n \in \mathbb{N}} \widehat{\mathbb{E}}_{\zeta_{n+1}, \zeta_n}^E \rho(\eta_t^1(0), \eta_t^2(0)) \leq \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)).$$

Using the translation invariance of the environment and $\rho \leq 1$ we obtain the estimate

$$\widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)) \leq \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_0} \frac{\widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(x), \eta_t^2(x))}{\rho(\eta(0), \xi(0))}.$$

Taking the supremum on the left hand side and integrating over time, we obtain that Assumption 1b indeed implies Assumption 1a.

Examples which satisfy Assumption 1a and 1b include:

- interacting particle systems in the so-called $M < \epsilon$;
- weakly interacting diffusions on a compact manifold;
- a system of ODEs which converges uniformly to its unique stationary configuration at sufficient speed.

Note that the third example is a deterministic environment and as such any form of condition which measures this convergence purely in probabilistic terms like the total variation distance is bound to fail.

3.2 Statement of the main theorem

The main result of this section is the following theorem, which tells us how the coupling property of the environment lifts to the environment process.

Theorem 3.1. *Let $f : \Omega \rightarrow \mathbb{R}$ with $\|f\| < \infty$.*

a) Under Assumption 1a, there exists a constant $C_a > 0$ so that

$$\sup_{\eta, \xi \in \Omega} \int_0^\infty |S_t^{EP} f(\eta) - S_t^{EP} f(\xi)| dt \leq C_a \|f\|.$$

b) Under Assumption 1b, there exists a constant $C_b > 0$ so that

$$\sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \int_0^\infty \frac{|S_t^{EP} f(\eta) - S_t^{EP} f(\xi)|}{\rho(\eta(x), \xi(x))} dt \leq C_b \|f\|.$$

This theorem encapsulates all the technical details and difficulties to obtain results, and subsection 3.5 is dedicated to its proof. In Section 3.4 we generalize this result to give more information about decay in time. Here we continue with results we can obtain using Theorem 3.1. Most results about the environment process just use part a) of the theorem, part b) is needed to obtain the CLT in Section 5 or more sophisticated deviation estimates in Section 6.

3.3 Existence of a unique ergodic measure and continuity in the rates

First of, the environment process, i.e. the environment as seen from the walker, is ergodic.

Lemma 3.2. *Under Assumption 1a the environment process has a unique ergodic probability measure μ^{EP} .*

Proof As E is compact, so is Ω , and therefore the space of stationary measures is non-empty. So we must just prove uniqueness.

Assume μ, ν are both stationary measures. Choose an arbitrary $f : \Omega \rightarrow \mathbb{R}$ with $\|f\| < \infty$. By Theorem 3.1,a,

$$\begin{aligned} T |\mu(f) - \nu(f)| &\leq \int \int \int_0^T |S_t^{EP} f(\eta) - S_t^{EP} f(\xi)| dt \mu(d\eta) \nu(d\xi) \\ &\leq \sup_{\eta, \xi \in \Omega} \int_0^\infty |S_t^{EP} f(\eta) - S_t^{EP} f(\xi)| dt < \infty. \end{aligned}$$

As T is arbitrary, $\mu(f) = \nu(f)$. As functions f with $\|f\| < \infty$ are dense in $\mathcal{C}(\Omega)$, there is at most one stationary probability measure. \square

It is of interest not only to know that the environment process has a unique ergodic measure μ^{EP} , but also to know how this measure depends on the rates α .

Theorem 3.3. *Under Assumption 1a, the unique ergodic measure μ_α^{EP} depends continuously on the rates α . For two transition rate functions α, α' , we have the following estimate:*

$$|\mu_\alpha^{EP}(f) - \mu_{\alpha'}^{EP}(f)| \leq \frac{C(\alpha)}{p(\alpha)} \|\alpha - \alpha'\|_0 \|f\|,$$

i.e.

$$(\alpha, f) \mapsto \mu_\alpha^{EP}(f)$$

is continuous in $\|\cdot\|_0 \times \|\cdot\|$. The functions $C(\alpha), p(\alpha)$ satisfy $C(\alpha) > 0, p(\alpha) \in]0, 1[$. In the case that the rates α do not depend on the environment, i.e. $\alpha(\eta, z) = \alpha(z)$, they are given by $p(\alpha) = 1$,

$$C(\alpha) = \int_0^\infty \sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)) dt.$$

As the proof is a variation of the proof of Theorem 3.1, it is delayed to the end of Section 3.5.

3.4 Speed of convergence to equilibrium in the environment process

We already know that under Assumption 1a the environment process has a unique ergodic distribution. However, we do not know at what speed the process converges to its unique stationary measure. Given the speed of convergence for the environment it is natural to believe that the environment process inherits that speed with some form of slowdown due to the additional self-interaction which is induced from the random walk. For example, if the original speed of convergence were exponential with a constant λ , then the environment process would also converge exponentially fast, but with a worse constant $\tilde{\lambda}$. This is indeed the case.

Theorem 3.4. *Let $\phi : [0, \infty[\rightarrow \mathbb{R}$ be a monotone increasing and continuous function satisfying $\phi(0) = 1$ and $\phi(s+t) \leq \phi(s)\phi(t)$.*

- a) *Suppose there exists a strong Markovian coupling $\widehat{\mathbb{P}}^E$ of the environment (with corresponding expectation $\widehat{\mathbb{E}}^E$) which satisfies*

$$\int_0^\infty \phi(t) t^d \sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)) dt < \infty.$$

Then there exists a constant $K_0 > 0$ and a decreasing function $C_a :]K_0, \infty[\rightarrow [0, \infty[$ so that for any $K > K_0$ and any $f : \Omega \rightarrow \mathbb{R}$ with $\|f\| < \infty$,

$$\sup_{\eta, \xi \in \Omega} \int_0^\infty \phi\left(\frac{t}{K}\right) |S_t^{EP} f(\eta) - S_t^{EP} f(\xi)| dt \leq C_a(K) \|f\|.$$

- b) *Suppose there exists a strong Markovian coupling $\widehat{\mathbb{P}}^E$ of the environment (with corresponding expectation $\widehat{\mathbb{E}}^E$) which satisfies*

$$\int_0^\infty \phi(t) t^d \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_0} \frac{\widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(x), \eta_t^2(x))}{\rho(\eta(0), \xi(0))} dt < \infty.$$

Then there exists a constant $K_0 > 0$ and a decreasing function $C_b :]K_0, \infty[\rightarrow [0, \infty[$ so that for any $K > K_0$ and any $f : \Omega \rightarrow \mathbb{R}$ with $\|f\| < \infty$,

$$\sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \int_0^\infty \phi\left(\frac{t}{K}\right) \frac{|S_t^{EP} f(\eta) - S_t^{EP} f(\xi)|}{\rho(\eta(x), \xi(x))} dt \leq C_b(K) \|f\|.$$

Canonical choices for ϕ are $\phi(t) = \exp(\lambda t)$ or $\phi(t) = (1+t)^\lambda$. In the first case, exponential decay of order λ of the environment is lifted to exponential decay of order λ/K for any $K > K_0$ in the environment process. In the second case, polynomial decay of order λ becomes polynomial decay of order $\lambda - d - \epsilon$ for any $0 < \epsilon \leq \lambda - d$.

3.5 Proofs of Theorem 3.1, 3.3 and 3.4

In this section we always assume that Assumption 1a holds.

We start with an outline of the idea of the proofs. We have a coupling of the environments (η_t^1, η_t^2) , which we extend to include two random walkers (X_t^1, X_t^2) , driven by their corresponding environment. We maximize the probability of both walkers performing the same jumps. Then Assumption 1a is sufficient to obtain a positive probability of both walkers staying together forever. If the walkers stay together, one just has to account for the difference in environments, but not the walkers as well. When the walkers split, the translation invariance allows for everything to be shifted that both walkers are back at the origin, and one can try again. After a geometric number of trials it is then guaranteed that the walkers stay together.

Proposition 3.5 (Coupling construction). *Given the coupling $\widehat{\mathbb{P}}_{\eta, \xi}^E$ of the environments, we extend it to a coupling $\widehat{\mathbb{P}}_{\eta, x; \xi, y}$. This coupling has the following properties:*

a) (Marginals) *The coupling supports two environments and corresponding random walkers:*

$$1) \widehat{\mathbb{P}}_{\eta, x; \xi, y}((\eta_t^1, X_t^1) \in \cdot) = \mathbb{P}_{\eta, x}((\eta_t, X_t) \in \cdot);$$

$$2) \widehat{\mathbb{P}}_{\eta, x; \xi, y}((\eta_t^2, X_t^2) \in \cdot) = \mathbb{P}_{\xi, y}((\eta_t, X_t) \in \cdot);$$

b) (Extension of $\widehat{\mathbb{P}}_{\eta, \xi}^E$) *The environments behave as under $\widehat{\mathbb{P}}^E$:*

$$\widehat{\mathbb{P}}_{\eta, x; \xi, y}((\eta_t^1, \eta_t^2) \in \cdot) = \widehat{\mathbb{P}}_{\eta, \xi}^E((\eta_t^1, \eta_t^2) \in \cdot);$$

c) (Coupling of the walkers) *X_t^1 and X_t^2 perform identical jumps as much as possible, the rate of performing a different jump is $\sum_{z \in \mathbb{Z}^d} \left| \alpha(\theta_{-X_t^1} \eta_t^1, z) - \alpha(\theta_{-X_t^2} \eta_t^2, z) \right|$;*

d) (Minimal and maximal walkers) *In addition to the environments η_t^1 and η_t^2 and random walkers X_t^1 and X_t^2 , the coupling supports minimal and maximal walkers Y_t^+ , Y_t^- as well. These two walkers have the following properties:*

1) $Y_t^- \leq X_t^1 - x, X_t^2 - y \leq Y_t^+$ $\widehat{\mathbb{P}}_{\eta, x; \xi, y}$ -a.s. (in dimension $d > 1$, this is to be interpreted coordinate-wise);

2) Y_t^+, Y_t^- are independent of η_t^1, η_t^2 ;

3) $\widehat{\mathbb{E}}_{\eta, x; \xi, y} Y_t^+ = t\gamma^+$ for some $\gamma^+ \in \mathbb{R}^d$;

4) $\widehat{\mathbb{E}}_{\eta, x; \xi, y} Y_t^- = t\gamma^-$ for some $\gamma^- \in \mathbb{R}^d$.

Proof The construction of this coupling $\widehat{\mathbb{P}}_{\eta, x; \xi, y}$ can be done in the following way: The environments behave according to $\widehat{\mathbb{P}}_{\eta, \xi}^E$, and $X_0^1 = x$, $X_0^2 = y$, $Y_0^+ = Y_0^- = 0$. For each $z \in \mathbb{Z}^d$ there is an independent Poisson clock with rate $\lambda_z := \sup_{\eta} \alpha(\eta, z)$. Whenever such a clock rings, draw an independent U from the uniform distribution in $[0, 1]$. If $U < \alpha(\theta_{-X_t^1} \eta_t^1, z) / \lambda_z$, X_t^1 performs a jump by z , analogue for X_t^2 . The

upper and lower walkers Y_t^+ and Y_t^- always jump on these clocks, however they jump by $\max(z, 0)$ or $\min(z, 0)$ respectively. N_t increases if

The properties of the coupling arise directly from the construction, except the last two. Those are a consequence from the first moment condition $\|\alpha\|_1 < \infty$. \square

Now we show how suitable estimates on the coupling speed of the environment translate to properties of the extended coupling.

Lemma 3.6.

$$\widehat{\mathbb{E}}_{\eta, x; \xi, y} \rho(\eta_t^1(X_t^1), \eta_t^2(X_t^2)) dt < (\|\gamma^+ - \gamma^-\|_\infty t + 1)^d \sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)).$$

Proof Denote with $R_t \subset \mathbb{Z}^d$ the set of sites x with $Y_t^- \leq x \leq Y_t^+$ (coordinate-wise). Then

$$\begin{aligned} & \sup_{\eta, \xi, x, y} \widehat{\mathbb{E}}_{\eta, x; \xi, y} \rho(\eta_t^1(X_t^1), \eta_t^2(X_t^2)) \\ & \leq \sup_{\eta, \xi, x, y} \widehat{\mathbb{E}}_{\eta, x; \xi, y} \sum_{z \in R_t} \rho(\eta_t^1(x+z), \eta_t^2(x+z)) \\ & \leq \widehat{\mathbb{E}} \left[\sum_{z \in R_t} 1 \right] \sup_{\eta, \xi, z} \widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(z), \eta_t^2(z)) \\ & \leq (\|\gamma^+ - \gamma^-\|_\infty t + 1)^d \sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)). \end{aligned}$$

\square

Lemma 3.7. Denote by $\tau := \inf\{t \geq 0 : X_t^1 \neq X_t^2\}$ the first time the two walkers are not at the same position. Under Assumption 1a,

$$\inf_{\eta, \xi \in \Omega} \widehat{\mathbb{P}}_{\eta, 0; \xi, 0}(\tau = \infty) > 0,$$

i.e., the walkers X^1 and X^2 never decouple with strictly positive probability.

Proof Both walkers start in the origin, therefore $\tau > 0$. The probability $\mathbb{P}(\exp((\lambda_t)) > T)$ that an exponential random variable with time dependent rate λ_t is bigger than T is given by $\exp(-\int_0^T \lambda_t dt)$. As the rate of decoupling is given by Proposition 3.5,c), we obtain

$$\begin{aligned} \widehat{\mathbb{P}}_{\eta, 0; \xi, 0}(\tau > T) &= \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \exp \left(- \int_0^T \sum_{z \in \mathbb{Z}^d} \left| \alpha(\theta_{-X_t^1} \eta_t^1, z) - \alpha(\theta_{-X_t^1} \eta_t^2, z) \right| dt \right) \\ &\geq \exp \left(- \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \int_0^T \sum_{z \in \mathbb{Z}^d} \left| \alpha(\theta_{-X_t^1} \eta_t^1, z) - \alpha(\theta_{-X_t^1} \eta_t^2, z) \right| dt \right). \end{aligned} \tag{5}$$

By telescoping over single site changes,

$$\begin{aligned}
& \widehat{\mathbb{E}}_{\eta,0;\xi,0} \sum_{z \in \mathbb{Z}^d} \left| \alpha(\theta_{-X_t^1} \eta_t^1, z) - \alpha(\theta_{-X_t^1} \eta_t^2, z) \right| \\
& \leq \widehat{\mathbb{E}}_{\eta,0;\xi,0} \sum_{z \in \mathbb{Z}^d} \sum_{x \in \mathbb{Z}^d} \rho(\eta_t^1(X_t^1 + x), \eta_t^2(X_t^1 + x)) \delta_{\alpha(\cdot, z)}(x) \\
& \leq \sup_{x \in \mathbb{Z}^d} \widehat{\mathbb{E}}_{\eta,0;\xi,0} \rho(\eta_t^1(X_t^1 + x), \eta_t^2(X_t^1 + x)) \|\alpha\| \\
& \leq \|\alpha\| (\|\gamma^+ - \gamma^-\|_\infty t + 1)^d \sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)),
\end{aligned}$$

where the last line follows from Lemma 3.6. With this estimate, we obtain

$$\begin{aligned}
\widehat{\mathbb{P}}_{\eta, \xi}(\tau = \infty) & \geq \exp \left(- \|\alpha\| \int_0^\infty (\|\gamma^+ - \gamma^-\|_\infty t + 1)^d \sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)) dt \right) \\
& > 0 \quad \text{uniformly in } \eta, \xi.
\end{aligned}$$

□

Proof of Theorem 3.1, part a) The idea of the proof is to use the coupling of Proposition 3.5: We wait until the walkers X_t^1 and X_t^2 , which are initially at the same position, decouple, and then restart everything and try again. By Lemma 3.7 there is a positive probability of never decoupling, so this scheme is successful. Using the time of decoupling τ (as in Lemma 3.7),

$$\begin{aligned}
& \int_0^T \left| \widehat{\mathbb{E}}_{\eta,0;\xi,0} \mathbb{1}_{t \geq \tau} \left(f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^2} \eta_t^2) \right) \right| dt \\
& = \int_0^T \left| \widehat{\mathbb{E}}_{\eta,0;\xi,0} \mathbb{1}_{t \geq \tau} \mathbb{E} \left[f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^2} \eta_t^2) \mid \mathfrak{F}_\tau \right] \right| dt \\
& \leq \int_0^T \widehat{\mathbb{E}}_{\eta,0;\xi,0} \mathbb{1}_{t \geq \tau} \left| S_{t-\tau}^{EP} f(\theta_{-X_\tau^1} \eta_\tau^1) - S_{t-\tau}^{EP} f(\theta_{-X_\tau^2} \eta_\tau^2) \right| dt \\
& = \widehat{\mathbb{E}}_{\eta,0;\xi,0} \int_0^{(T-\tau) \vee 0} \left| S_t^{EP} f(\theta_{-X_\tau^1} \eta_\tau^1) - S_t^{EP} f(\theta_{-X_\tau^2} \eta_\tau^2) \right| dt \tag{6}
\end{aligned}$$

$$\leq \widehat{\mathbb{P}}_{\eta,0;\xi,0}(\tau < \infty) \sup_{\eta, \xi \in \Omega} \int_0^T \left| S_t^{EP} f(\eta) - S_t^{EP} f(\xi) \right| dt. \tag{7}$$

And therefore

$$\begin{aligned}
& \int_0^T |S_t^{EP} f(\eta) - S_t^{EP} f(\xi)| dt \\
&= \int_0^T \left| \widehat{\mathbb{E}}_{\eta,0;\xi,0} f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^2} \eta_t^2) \right| dt \\
&\leq \int_0^T \widehat{\mathbb{E}}_{\eta,0;\xi,0} \mathbb{1}_{t < \tau} \left| f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^2} \eta_t^2) \right| dt \\
&\quad + \widehat{\mathbb{P}}_{\eta,0;\xi,0}(\tau < \infty) \sup_{\eta, \xi \in \Omega} \int_0^T |S_t^{EP} f(\eta) - S_t^{EP} f(\xi)| dt \tag{8}
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^\infty \widehat{\mathbb{E}}_{\eta,0;\xi,0} \left| f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^2} \eta_t^2) \right| dt \\
&\quad + \widehat{\mathbb{P}}_{\eta,0;\xi,0}(\tau < \infty) \sup_{\eta, \xi \in \Omega} \int_0^T |S_t^{EP} f(\eta) - S_t^{EP} f(\xi)| dt, \tag{9}
\end{aligned}$$

which gives us the upper bound

$$\begin{aligned}
&\sup_{\eta, \xi \in \Omega} \int_0^\infty |S_t^{EP} f(\eta) - S_t^{EP} f(\xi)| dt \\
&\leq \left(\inf_{\eta, \xi \in \Omega} \widehat{\mathbb{P}}_{\eta,0;\xi,0}(\tau = \infty) \right)^{-1} \sup_{\eta, \xi \in \Omega} \int_0^\infty \widehat{\mathbb{E}}_{\eta,0;\xi,0} \left| f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^2} \eta_t^2) \right| dt. \tag{10}
\end{aligned}$$

To show that the last integral is finite, we telescope over single site changes, and get

$$\begin{aligned}
&\int_0^\infty \widehat{\mathbb{E}}_{\eta,0;\xi,0} \left| f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^2} \eta_t^2) \right| dt \\
&\leq \int_0^\infty \widehat{\mathbb{E}}_{\eta,0;\xi,0} \sum_{x \in \mathbb{Z}^d} \rho(\eta_t^1(x + X_t^1), \eta_t^2(x + X_t^1)) \delta_f(x) dt \\
&\leq \|f\| \sup_{\eta, \xi, x} \int_0^\infty \widehat{\mathbb{E}}_{\eta,0;\xi,0} \rho(\eta_t^1(x + X_t^1), \eta_t^2(x + X_t^1)) dt,
\end{aligned}$$

which is finite by Lemma 3.6 and Assumption 1a). Choosing

$$C_a = \left(\inf_{\eta, \xi \in \Omega} \widehat{\mathbb{P}}_{\eta,0;\xi,0}(\tau = \infty) \right)^{-1} \sup_{\eta, \xi, x} \int_0^\infty \widehat{\mathbb{E}}_{\eta,0;\xi,0} \rho(\eta_t^1(x + X_t^1), \eta_t^2(x + X_t^1)) dt \tag{11}$$

completes the proof. \square

To prove part b) of the theorem, we need the following analogue to Lemma 3.7 using Assumption 1b).

Lemma 3.8. *Under Assumption 1b, for every site-weight function $w : \mathbb{Z}^d \rightarrow [0, \infty[$ with $\|w\|_1 := \sum_x w(x) < \infty$, we have*

$$\sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \int_0^\infty \sum_{y \in \mathbb{Z}^d} w(y) \frac{\widehat{\mathbb{E}}_{\eta, \xi} \rho(\eta_t^1(y + X_t^1), \eta_t^2(y + X_t^1))}{\rho(\eta(x), \xi(x))} dt \leq \text{const} \cdot \|w\|_1.$$

Proof Denote with $R_t \subset \mathbb{Z}^d$ the set of sites whose j th coordinate lies between $Y_t^{j,-}$ and $Y_t^{j,+}$. Then,

$$\begin{aligned} & \sum_{y \in \mathbb{Z}^d} w(y) \widehat{\mathbb{E}}_{\eta, \xi} \rho(\eta_t^1(y + X_t^1), \eta_t^2(y + X_t^1)) \\ &= \sum_{y \in \mathbb{Z}^d} \widehat{\mathbb{E}}_{\eta, \xi} w(y - X_t^1) \rho(\eta_t^1(y), \eta_t^2(y)) \\ &\leq \sum_{y \in \mathbb{Z}^d} \widehat{\mathbb{E}}_{\eta, \xi} \sum_{z \in R_t} w(y - z) \rho(\eta_t^1(y), \eta_t^2(y)) \\ &= \sum_{y \in \mathbb{Z}^d} \mathbb{E} \left[\sum_{z \in R_t} w(y - z) \right] \widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(y), \eta_t^2(y)) \end{aligned}$$

by independence of R_t and (η_t^1, η_t^2) . Therewith,

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \int_0^\infty \sum_{y \in \mathbb{Z}^d} w(y) \frac{\widehat{\mathbb{E}}_{\eta, \xi} \rho(\eta_t^1(y + X_t^1), \eta_t^2(y + X_t^1))}{\rho(\eta(x), \xi(x))} dt \\ &\leq \int_0^\infty \sum_{y \in \mathbb{Z}^d} \mathbb{E} \left[\sum_{z \in R_t} w(y - z) \right] \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \widehat{\mathbb{E}}_{\eta, \xi}^E \frac{\rho(\eta_t^1(y), \eta_t^2(y))}{\rho(\eta(x), \xi(x))} dt. \end{aligned}$$

Note that by translation invariance the right part is equal to

$$\sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_0} \widehat{\mathbb{E}}_{\eta, \xi}^E \frac{\rho(\eta_t^1(x), \eta_t^2(x))}{\rho(\eta(0), \xi(0))}$$

and by construction of R_t and Proposition 3.5.d ,

$$\begin{aligned} \sum_{y \in \mathbb{Z}^d} \mathbb{E} \left[\sum_{z \in R_t} w(y - z) \right] &= \mathbb{E} \left[\sum_{z \in R_t} 1 \right] \|w\|_1 = \prod_{j=1}^d (\gamma^{j,+} t - \gamma^{j,-} t + 1) \|w\|_1 \\ &\leq c_w (t^d + 1) \end{aligned}$$

for some suitable $c_w > 0$. Therefore Assumption 1b completes the proof. \square

Proof of Theorem 3.1, part b) Let $\tau := \inf\{t \geq 0 : X_t^1 \neq X_t^2\}$. Then we split the

integration at τ :

$$\begin{aligned}
& \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \int_0^\infty \frac{|S_t^{EP} f(\eta) - S_t^{EP} f(\xi)|}{\rho(\eta(x), \xi(x))} dt \\
& \leq \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \int_0^\infty \frac{|\widehat{\mathbb{E}}_{\eta, \xi} \mathbb{1}_{\tau > t} (f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^1} \eta_t^2))|}{\rho(\eta(x), \xi(x))} dt \\
& \quad + \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \int_0^\infty \frac{|\widehat{\mathbb{E}}_{\eta, \xi} \mathbb{1}_{\tau \leq t} (f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^2} \eta_t^2))|}{\rho(\eta(x), \xi(x))} dt
\end{aligned}$$

We estimate the first term by moving the expectation out of the absolute value and forgetting the restriction to $\tau > t$:

$$\sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \int_0^\infty \sum_{y \in \mathbb{Z}^d} \delta_f(y) \frac{\widehat{\mathbb{E}}_{\eta, \xi} \rho(\eta_t^1(y + X_t^1), \eta_t^2(y + X_t^1))}{\rho(\eta(x), \xi(x))} dt.$$

By Lemma 3.8 with $w = \delta_f$, this is bounded by some constant times $\|f\|$. For the second term we start by using the Markov property:

$$\begin{aligned}
& \int_0^\infty \left| \widehat{\mathbb{E}}_{\eta, \xi} \mathbb{1}_{\tau \leq t} (f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^2} \eta_t^2)) \right| dt \\
& = \int_0^\infty \left| \widehat{\mathbb{E}}_{\eta, \xi} \mathbb{1}_{\tau \leq t} (S_{t-\tau}^{EP} f(\theta_{-X_\tau^1} \eta_\tau^1) - S_{t-\tau}^{EP} f(\theta_{-X_\tau^2} \eta_\tau^2)) \right| dt \\
& \leq \widehat{\mathbb{E}}_{\eta, \xi} \mathbb{1}_{\tau < \infty} \int_\tau^\infty |S_{t-\tau}^{EP} f(\theta_{-X_\tau^1} \eta_\tau^1) - S_{t-\tau}^{EP} f(\theta_{-X_\tau^2} \eta_\tau^2)| dt \tag{12} \\
& \leq \widehat{\mathbb{P}}_{\eta, \xi}(\tau < \infty) \sup_{\eta, \xi \in \Omega} \int_0^\infty |S_t^{EP} f(\eta) - S_t^{EP} f(\xi)| dt. \tag{13}
\end{aligned}$$

By part a) of Theorem 3.1 the integral part is uniformly bounded by $C_a \|f\|$. So what remains to complete the proof is to show that

$$\sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \frac{\widehat{\mathbb{P}}_{\eta, \xi}(\tau < \infty)}{\rho(\eta(x), \xi(x))} < \infty. \tag{14}$$

To do so we first use the same idea as in the proof of Lemma 3.7 to obtain

$$\begin{aligned}
& \widehat{\mathbb{P}}_{\eta, \xi}(\tau < \infty) \\
& = 1 - \exp \left(- \int_0^\infty \widehat{\mathbb{E}}_{\eta, \xi} \sum_{z \in \mathbb{Z}^d} \left| \alpha(\theta_{-X_t^1} \eta_t^1, z) - \alpha(\theta_{-X_t^1} \eta_t^2, z) \right| dt \right) \\
& \leq \int_0^\infty \widehat{\mathbb{E}}_{\eta, \xi} \sum_{z \in \mathbb{Z}^d} \left| \alpha(\theta_{-X_t^1} \eta_t^1, z) - \alpha(\theta_{-X_t^1} \eta_t^2, z) \right| dt \\
& \leq \int_0^\infty \sum_{y \in \mathbb{Z}^d} w_\alpha(y) \widehat{\mathbb{E}}_{\eta, \xi} \rho(\eta_t^1(y + X_t^1), \eta_t^2(y + X_t^1)) dt
\end{aligned}$$

with

$$w_\alpha(x) := \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \sum_{z \in \mathbb{Z}^d} |\alpha(\eta, z) - \alpha(\xi, z)|$$

and $\sum_{x \in \mathbb{Z}^d} w_\alpha(x) < \infty$. So we get

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \frac{\widehat{\mathbb{P}}_{\eta, \xi}(\tau < \infty)}{\rho(\eta(x), \xi(x))} \\ & \leq \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \int_0^\infty \sum_{y \in \mathbb{Z}^d} w_\alpha(y) \frac{\widehat{\mathbb{E}}_{\eta, \xi} \rho(\eta_t^1(y + X_t^1), \eta_t^2(y + X_t^1))}{\rho(\eta(x), \xi(x))} dt, \end{aligned}$$

and Lemma 3.8 completes the proof, where C_b is the combination of the various factors in front of $\|f\|$. \square

Proof of Theorem 3.3 Let α, α' be two different transition rates. The goal is to show that

$$|\mu_\alpha^{EP}(f) - \mu_{\alpha'}^{EP}(f)| \leq C \|f\|$$

for all $f : \Omega \rightarrow \mathbb{R}$ with $\|f\| < \infty$.

The idea is now to use a coupling $\widehat{\mathbb{P}}$ similar to the one in Proposition 3.5. The coupling contains as objects two copies of the environment, η^1 and η^2 , and three random walks, X^1, X^{12} and X^2 . The random walk X^1 moves on the environment η^1 with rates α , and correspondingly the random walk X^2 moves on η^2 with rates α' . The mixed walker X^{12} moves on the environment η^2 as well, but according to the rates α . The walkers X^1, X^2 will perform the same jumps as X^{12} with maximal probability. The environments we couple utilizing the Markovian coupling provided by Assumption 1a. We only consider the case where all three walkers start at the origin. We denote by $S_t^{EP,1}, S_t^{EP,2}$ the semigroups of the environment process which correspond to the rates α and α' . Let $\tau := \inf\{t \geq 0 : X_t^1 \neq X_t^{12} \text{ or } X_t^{12} \neq X_t^2\}$.

$$\begin{aligned} & S_t^{EP,1} f(\eta) - S_t^{EP,2} f(\xi) \\ & = \widehat{\mathbb{E}}_{\eta, \xi} \left(f(\theta_{-X_t^1 \eta_t^1}) - f(\theta_{-X_t^2 \eta_t^2}) \right) \\ & = \widehat{\mathbb{E}}_{\eta, \xi} \mathbb{1}_{\tau > t} \left(f(\theta_{-X_t^1 \eta_t^1}) - f(\theta_{-X_t^2 \eta_t^2}) \right) \\ & \quad + \widehat{\mathbb{E}}_{\eta, \xi} \mathbb{1}_{\tau \leq t} \left(f(\theta_{-X_t^1 \eta_t^1}) - f(\theta_{-X_t^2 \eta_t^2}) \right) \\ & = \widehat{\mathbb{E}}_{\eta, \xi} \mathbb{1}_{\tau > t} \left(f(\theta_{-X_t^1 \eta_t^1}) - f(\theta_{-X_t^2 \eta_t^2}) \right) \\ & \quad + \widehat{\mathbb{E}}_{\eta, \xi} \mathbb{1}_{\tau \leq t} \left(S_{t-\tau}^{EP,1} f(\theta_{-X_\tau^1 \eta_\tau^1}) - S_{t-\tau}^{EP,2} f(\theta_{-X_\tau^2 \eta_\tau^2}) \right). \end{aligned}$$

Therefore,

$$\begin{aligned}
\Psi(T) &:= \sup_{0 \leq T' \leq T} \sup_{\eta, \xi \in \Omega} \int_0^{T'} S_t^{EP,1} f(\eta) - S_t^{EP,2} f(\xi) dt \\
&\leq \sup_{0 \leq T' \leq T} \sup_{\eta, \xi \in \Omega} \int_0^{T'} \widehat{\mathbb{E}}_{\eta, \xi} \mathbb{1}_{\tau > t} \left(f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^1} \eta_t^2) \right) \\
&\quad + \widehat{\mathbb{E}}_{\eta, \xi} \mathbb{1}_{\tau \leq t} \sup_{\eta, \xi \in \Omega} \left(S_{t-\tau}^{EP,1} f(\eta) - S_{t-\tau}^{EP,2} f(\xi) \right) dt \\
&\leq \sup_{0 \leq T' \leq T} \sup_{\eta, \xi \in \Omega} \left(\widehat{\mathbb{E}}_{\eta, \xi} \int_0^T f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^1} \eta_t^2) dt + \mathbb{1}_{\tau \leq T'} \Psi(T' - \tau) \right) \\
&\leq \sup_{\eta, \xi \in \Omega} \left(\widehat{\mathbb{E}}_{\eta, \xi} \int_0^\infty f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^1} \eta_t^2) dt + \mathbb{1}_{\tau \leq T} \Psi(T - \tau) \right). \quad (15)
\end{aligned}$$

We will now exploit this recursive bound on Ψ .

Lemma 3.9. *Let $\tau_1 := \inf\{t \geq 0 : X_t^1 \neq X_t^{12}\}$ and $\tau_2 := \inf\{t \geq 0 : X_t^{12} \neq X_t^2\}$. Set*

$$\begin{aligned}
\beta &:= \sum_{z \in \mathbb{Z}^d} \sup_{\eta \in \mathbb{Z}^d} |\alpha(\eta, z) - \alpha'(\eta, z)|, \\
p(\alpha) &:= \inf_{\eta, \xi \in \Omega} \widehat{\mathbb{P}}_{\eta, \xi}(\tau_1 = \infty), \\
C(\alpha) &:= \int_0^\infty (\|\gamma^+(\alpha) - \gamma^-(\alpha)\|_\infty t + 1)^d \sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)) dt,
\end{aligned}$$

where $\gamma^+(\alpha), \gamma^-(\alpha)$ are as in Proposition 3.5 for the rates α .

Let $Y \in \{0, 1\}$ be Bernoulli with parameter $p(\alpha)$ and Y' exponentially distributed with parameter β . Let Y_1, Y_2, \dots be iid. copies of $Y \cdot Y'$ and $N(T) := \inf\{N \geq 0 : \sum_{n=1}^N Y_n > T\}$. Then

$$\Psi(T) \leq C(\alpha) \mathbb{I} f \mathbb{I} \mathbb{E} N(T)$$

Proof By construction of the coupling, τ_2 stochastically dominates Y' . As we have $\tau = \tau_1 \wedge \tau_2$ it follows that $\tau \succeq Y_1$. Using this fact together with the monotonicity of Ψ in (15),

$$\begin{aligned}
\Psi(T) &\leq \sup_{\eta, \xi \in \Omega} \left(\widehat{\mathbb{E}}_{\eta, \xi} \int_0^\infty f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^1} \eta_t^2) dt + \mathbb{1}_{\tau \leq T} \Psi(T - \tau) \right) \\
&\leq \sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, \xi} \int_0^\infty f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^1} \eta_t^2) dt + \mathbb{E} \mathbb{1}_{Y_1 \leq T} \Psi(T - Y_1).
\end{aligned}$$

As $p(\alpha) > 0$ by Lemma 3.7 we can iterate this estimate until it terminates after $N(T)$ steps. Therefore we obtain

$$\Psi(T) \leq \mathbb{E} N(T) \sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, \xi} \int_0^\infty f(\theta_{-X_t^1} \eta_t^1) - f(\theta_{-X_t^1} \eta_t^2) dt.$$

The integral is estimated by telescoping over single site changes and Lemma 3.6 in the usual way, yielding

$$\Psi(T) \leq C(\alpha) \mathbb{I} f \mathbb{I} \mathbb{E} N(T).$$

□

To finally come back to the original question of continuity,

$$\begin{aligned} |\mu_\alpha^{EP}(f) - \mu_{\alpha'}^{EP}(f)| &= \frac{1}{T} \left| \int \int \int_0^T S_t^{EP,1} f(\eta) - S_t^{EP,2} f(\xi) dt \mu_\alpha^{EP}(d\eta) \mu_{\alpha'}^{EP}(d\xi) \right| \\ &\leq \frac{1}{T} \Psi(T) \leq \frac{1}{T} \mathbb{E} N(T) C(\alpha) \mathbb{I} f \mathbb{I} \\ &\xrightarrow{T \rightarrow \infty} \frac{1}{\mathbb{E} Y Y'} C(\alpha) \mathbb{I} f \mathbb{I} \\ &= \frac{C(\alpha)}{p(\alpha)} \sum_{z \in \mathbb{Z}^d} \sup_{\eta \in \mathbb{Z}^d} |\alpha(\eta, z) - \alpha'(\eta, z)| \mathbb{I} f \mathbb{I}. \end{aligned}$$

By sending α' to α , the right hand side tends to 0 so that the ergodic measure of the environment process is indeed continuous in the rates α . It is also interesting to note that both $p(\alpha)$ and $C(\alpha)$ are rather explicit given the original coupling of the environment. Notably when $\alpha(\eta, z) = \alpha(z)$, i.e. the rates do not depend on the environment, $p(\alpha) = 1$ and $C(\alpha) = \int_0^\infty \sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)) dt$. □

Proof of Theorem 3.4 The proof of this theorem is mostly identical to the proof of Theorem 3.1. Hence instead of copying the proof, we just state where details differ.

A first fact is that the conditions for a) and b) imply Assumptions 1a) and b). In the adaptation of the proof for part a), in most lines it suffices to add a $\phi\left(\frac{t}{K}\right)$ to the integrals. However, in line (6), we use

$$\phi\left(\frac{t}{K}\right) \leq \phi\left(\frac{t-\tau}{K}\right) \phi\left(\frac{\tau}{K}\right) \quad (16)$$

to obtain the estimate

$$\widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \phi\left(\frac{\tau}{K}\right) \int_0^{(T-\tau) \vee 0} \phi(t) |S_t^{EP} f(\theta_{-X_\tau^1} \eta_\tau^1) - S_t^{EP} f(\theta_{-X_\tau^2} \eta_\tau^2)| dt$$

instead. Thereby in lines (7), (8) and (9) we have to change $\widehat{\mathbb{P}}_{\eta, 0; \xi, 0}(\tau < \infty)$ to $\widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \phi\left(\frac{\tau}{K}\right) \mathbb{1}_{\tau < \infty}$. This change then leads to the replacement of

$$\inf_{\eta, \xi \in \Omega} \widehat{\mathbb{P}}_{\eta, 0; \xi, 0}(\tau = \infty)$$

by the term

$$1 - \sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \phi\left(\frac{\tau}{K}\right) \mathbb{1}_{\tau < \infty}$$

in the lines (10) and (11) (where naturally C_a becomes $C_a(K)$). So all we have to prove that for sufficiently big K

$$\sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \phi \left(\frac{\tau}{K} \right) \mathbb{1}_{\tau < \infty} < 1.$$

In a first step, we show that

$$\sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \phi(\tau) \mathbb{1}_{\tau < \infty} < \infty.$$

As we already saw in the proof of Lemma 3.7, we can view the event of decoupling as the first jump of a Poisson process with time-dependent and random rates (equation (5)). Hence we have

$$\begin{aligned} \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \phi(\tau) \mathbb{1}_{\tau < \infty} &= \int_0^\infty \phi(t) d\widehat{\mathbb{P}}_{\eta, 0; \xi, 0}(\tau > t) \\ &= \int_0^\infty \phi(t) \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \sum_{z \in \mathbb{Z}^d} \left| \alpha(\theta_{-X_t^1} \eta_t^1, z) - \alpha(\theta_{-X_t^1} \eta_t^2, z) \right| \\ &\quad \cdot \exp \left(- \int_0^t \sum_{z \in \mathbb{Z}^d} \left| \alpha(\theta_{-X_s^1} \eta_s^1, z) - \alpha(\theta_{-X_s^1} \eta_s^2, z) \right| ds \right) dt \\ &\leq \int_0^\infty \phi(t) \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \sum_{z \in \mathbb{Z}^d} \left| \alpha(\theta_{-X_t^1} \eta_t^1, z) - \alpha(\theta_{-X_t^1} \eta_t^2, z) \right| dt. \end{aligned}$$

By telescoping over single site discrepancies and using Lemma 3.6, this is less than

$$\int_0^\infty \phi(t) (\|\gamma^+ - \gamma^-\|_\infty + 1)^{d_t} \sup_{\eta, \xi \in \Omega} \widehat{\mathbb{E}}_{\eta, \xi}^E \rho(\eta_t^1(0), \eta_t^2(0)) dt < \infty$$

by assumption. Since $\phi(t/K)$ decreases to 1 as $K \rightarrow \infty$, monotone convergence implies

$$\lim_{K \rightarrow \infty} \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \phi \left(\frac{\tau}{K} \right) \mathbb{1}_{\tau < \infty} = \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \mathbb{1}_{\tau < \infty} < 1$$

by Lemma 3.7. Consequently, there exists a $K_0 \geq 0$ such that for all $K > K_0$

$$\widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \phi \left(\frac{\tau}{K} \right) \mathbb{1}_{\tau < \infty} < 1.$$

This completes the adaptation of part a).

The adaptation of the proof of part b) follows the same scheme, where we add the term $\phi \left(\frac{t}{K} \right)$ to all integrals. Note that this gives a version of Lemma 3.8 as well. Then, in line (12) we use (16) again and then have to replace $\widehat{\mathbb{P}}_{\eta, 0; \xi, 0}(\tau < \infty)$ by $\widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \phi \left(\frac{\tau}{K} \right) \mathbb{1}_{\tau < \infty}$ in lines (13) and (14). To estimate (14), we use

$$\begin{aligned} \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \phi \left(\frac{\tau}{K} \right) \mathbb{1}_{\tau < \infty} &\leq \int_0^\infty \phi \left(\frac{t}{K} \right) \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \sum_{z \in \mathbb{Z}^d} \left| \alpha(\theta_{-X_t^1} \eta_t^1, z) - \alpha(\theta_{-X_t^1} \eta_t^2, z) \right| dt \\ &\leq \int_0^\infty \phi \left(\frac{t}{K} \right) \sum_{y \in \mathbb{Z}^d} w_\alpha \widehat{\mathbb{E}}_{\eta, 0; \xi, 0} \rho(\eta_t^1(y + X_t^1), \eta_t^2(y + X_t^1)) dt \end{aligned}$$

with w_α as in the original proof. Therefore

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \frac{\widehat{\mathbb{E}}_{\eta, \xi} \phi\left(\frac{\tau}{K}\right) \inf_{\tau < \infty}}{\rho(\eta(x), \xi(x))} \\ & \leq \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \int_0^\infty \phi\left(\frac{t}{K}\right) \sum_{y \in \mathbb{Z}^d} w_\alpha(y) \frac{\widehat{\mathbb{E}}_{\eta, \xi} \rho(\eta_t^1(y + X_t^1), \eta_t^2(y + X_t^1))}{\rho(\eta(x), \xi(x))} dt, \end{aligned}$$

which is finite by Lemma 3.8. \square

4 Law of Large Numbers and Einstein Relation

4.1 Law of Large Numbers for the position of the walker

Theorem 4.1. *Under Assumption 1a,*

$$\lim_{T \rightarrow \infty} \frac{X_T}{T} = \int \sum_{z \in \mathbb{Z}^d} z \alpha(\eta, z) \mu^{EP}(d\eta)$$

in $L_1(\mathbb{P}_{\nu,0})$ and $\mathbb{P}_{\nu,0}$ -a.s. for any probability measure ν on Ω .

Proof First we assume that the only shifts the environment process $\eta_t^{EP} = \theta_{-X_t} \eta_t$ performs are induced by jumps of the walker X_t . Assumption 1a does not prohibit the environment to perform shifts itself, but the argument comes more natural when it does not.

Let $F_z : D([0, 1], \Omega) \rightarrow \mathbb{R}$, $z \in \mathbb{Z}^d$ count the number of shifts of size z a piece of trajectory performs in the interval $[0, 1]$, i.e.

$$F_z(\eta_{[t, t+1]}^{EP}) = \sum_{s \in [0, 1]} \mathbb{1}_{\theta_{-z} \eta_{t+s}^{EP} = \eta_{t+s-}^{EP}}.$$

With $F := \sum_{z \in \mathbb{Z}^d} z F_z$, which is well-defined and in $L^1(\mu^{EP})$ because of the rate condition $\|\alpha\|_1 < \infty$, we then have for any integer $T > 0$

$$X_T - X_0 = \sum_{n=1}^T F(\eta_{[n-1, n]}^{EP}).$$

The ergodic theorem then implies

$$\lim_{T \rightarrow \infty} \frac{X_T - X_0}{T} = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=1}^T F(\eta_{[n-1, n]}^{EP}) = \mu^{EP}(F).$$

The same is true for non-integer T , as we simply use the fact that $X_T - X_{\lfloor T \rfloor}$ has bounded expectation. Since

$$\begin{aligned}\mu^{EP}(F) &= \int \int_0^1 \mathbb{E}_{\eta,0} \sum_{z \in \mathbb{Z}^d} z \alpha(\theta_{-X_t} \eta_t, z) dt \mu^{EP}(d\eta) \\ &= \int \sum_{z \in \mathbb{Z}^d} z \alpha(\eta, z) \mu^{EP}(d\eta),\end{aligned}$$

the claim is proven for μ^{EP} if the environment performs no shifts.

In the case that the environment does perform shifts, we introduce an additional auxiliary counting processes N_t^z on \mathbb{N} , $z \in \mathbb{Z}^d$, where N_t^z increases by one whenever $X_t - X_{t-} = z$ and N_t^z jumps to 0 at rate one to make it stationary. Then F_z counts the number of increments of N_t^z instead of the number of shifts. The rest of the proof is the same.

To extend the result to an arbitrary probability measure ν we use a coupling argument. We look at $|X_T^1 - X_T^2|$ under $\widehat{\mathbb{P}}_{\nu,0;\mu^{EP},0}$, where we slightly modify the coupling from Proposition 3.5 so that when the two walkers decouple at a time τ we restart the coupling of the environment. This restart is done on $\theta_{-X_\tau^1} \eta_\tau^1, \theta_{-X_\tau^2} \eta_\tau^2$ so that after the decoupling it acts as if both walkers are back at the origin. Then, by Lemma 3.7, there is at least probability $p > 0$ to never decouple initially or after a decoupling. So there is at most a geometric number N of decoupling events at τ_1, \dots, τ_N , and only at decoupling events the two walkers perform different jumps (one jumps, the other does not). Hence

$$\widehat{\mathbb{E}}_{\nu,0;\mu^{EP},0} |X_T^1 - X_T^2| \leq \widehat{\mathbb{E}}_{\nu,0;\mu^{EP},0} \sum_{n=1}^N (|X_{\tau_n}^1 - X_{\tau_n-}^1| + |X_{\tau_n}^2 - X_{\tau_n-}^2|),$$

which converges to 0 in L^1 and almost surely when divided by T and sending T to infinity. \square

4.2 Weak interaction with the environment and an Einstein Relation

It can be interesting to consider random walks which are only weakly affected by an external influence. In the context we study here, that corresponds to rates α which are only weakly dependent on the state η of the environment.

To study this weak interaction, let $\alpha_\epsilon : \Omega \times \mathbb{Z}^d \rightarrow [0, \infty[$, $0 \leq \epsilon$ be rates for the walker, satisfying the conditions from Section 2.3. Furthermore we assume that $\epsilon \mapsto \alpha_\epsilon(\eta, z)$ is differentiable in 0 and $\alpha_0(\eta, z) = \alpha_0(z)$, that is for $\epsilon = 0$ the walker is independent of the environment. Finally we assume that $\|\alpha_\epsilon - \alpha_0\|_1 \leq C\epsilon$ for some constant $C > 0$.

Denote with v_ϵ the asymptotic speed of the walker with rates α_ϵ .

Theorem 4.2. *The change of the ergodic measure of the environment process from μ^E , the ergodic measure of the environment, to μ_ϵ^{EP} , the ergodic measure corresponding to*

α_ϵ , is a second order influence on the speed v_ϵ of the walker:

$$\left| v_\epsilon - \int \sum_{z \in \mathbb{Z}^d} z \alpha_\epsilon(\eta, z) \mu^E(d\eta) \right| \leq C' \epsilon^2,$$

where C' is a constant independent of α_ϵ , $\epsilon \geq 0$.

Proof By Theorem 4.1,

$$v_\epsilon = \int \sum_{z \in \mathbb{Z}^d} z \alpha_\epsilon(\eta, z) \mu_\epsilon^{EP}(d\eta).$$

Note that $\mu_0^{EP} = \mu^E$. By Theorem 3.3,

$$\begin{aligned} & \left| \int \sum_{z \in \mathbb{Z}^d} z \alpha_\epsilon(\eta, z) \mu_\epsilon^{EP}(d\eta) - \int \sum_{z \in \mathbb{Z}^d} z \alpha_\epsilon(\eta, z) \mu_0^{EP}(d\eta) \right| \\ & \leq C(\alpha_0) \|\alpha_\epsilon - \alpha_0\|_0 \left\| \sum_{z \in \mathbb{Z}^d} z \alpha_\epsilon(\cdot, z) \right\| \end{aligned}$$

and $C(\alpha_0)$ is in fact independent of α_0 as those rates are independent of the environment. The estimates $\|\alpha_\epsilon - \alpha_0\|_0 \leq \|\alpha_\epsilon - \alpha_0\|_1 \leq C\epsilon$ and

$$\left\| \sum_{z \in \mathbb{Z}^d} z \alpha_\epsilon(\cdot, z) \right\| \leq \|\alpha_\epsilon\|_1 = \|\alpha_\epsilon - \alpha_0\|_1 \leq C\epsilon$$

complete the proof. \square

An *Einstein Relation* is said to hold if the change of speed of the walker because of the external influence is equal to the diffusion constant of the walker in the limit of the influence going to 0, or more precisely

$$\lim_{\epsilon \rightarrow 0} \frac{v_\epsilon - v_0}{\epsilon} = \sigma_0^2 \quad (17)$$

with $\sigma_0^2 = \sum_{z \in \mathbb{Z}} z^2 \alpha_0(z)$. Note that the typical formulation is in the case $v_0 = 0$.

Corollary 4.3. *The Einstein Relation (17) holds iff*

$$\sum_{z \in \mathbb{Z}} z \int \alpha'_0(\eta, z) \mu^E(d\eta) = \sum_{z \in \mathbb{Z}} z^2 \alpha_0(z),$$

where $\alpha'_0(\eta, z)$ is the derivative in ϵ .

Especially, the Einstein Relation holds if

$$\int \alpha'_0(\eta, z) \mu^E(d\eta) = z \alpha_0(z) \quad \forall z \in \mathbb{Z}.$$

Proof By Theorem 4.2,

$$v_\epsilon = \int \sum_{z \in \mathbb{Z}^d} z \alpha_\epsilon(\eta, z) \mu^E(d\eta) + O(\epsilon^2)$$

and hence

$$\lim_{\epsilon \rightarrow 0} \frac{v_\epsilon - v_0}{\epsilon} = \sum_{z \in \mathbb{Z}} z \int \alpha'_0(\eta, z) \mu^E(d\eta). \quad \square$$

5 Central Limit Theorem

Before starting with the functional central limit theorem for the random walk (X_t) in its random dynamic environment let us quickly discuss the CLT for additive functionals of the environment process.

Let us assume that $f : \Omega \rightarrow \mathbb{R}$ with $\|f\| < \infty$ and $\mu^{EP}(f) = 0$. Then

$$(L^{EP})^{-1} f(\eta) = - \int_0^\infty S_t^{EP} f(\eta) dt,$$

where Theorem 3.1 guarantees that the right hand side is well defined. Hence standard arguments provide a functional CLT for additive functionals $\int_0^T f(\eta_t^{EP}) dt$ in this context.

More work is necessary for a CLT of the position of the walker. The standard approach is to use the fact that the difference between the position of the walker X_T and the cumulate rates $\int_0^T \sum_{z \in \mathbb{Z}^d} z \alpha(\theta_{-X_t} \eta_t, z) dt$ is a martingale. One can prove a CLT for this martingale, and together with the CLT for the drift-adjusted integral obtain a CLT for the position. However, one has to take care that the variances do not annihilate, and generally there is no explicit formulation of the variance. As the martingales used in the appendix provide an alternative method for obtaining a CLT which does include an explicit formulation of the variance, we explore that approach.

5.1 A second assumption on the environment

There is one additional assumption on the environment required to easily use the martingales (and the moment estimates they provide).

Assumption 2 *There exists a constant $R^E \geq 0$ such that for any $f : \Omega \rightarrow \mathbb{R}$ with $\|f\| < \infty$,*

$$\overline{L}^E(f - f(\eta))^2(\eta) := \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} S_\epsilon^E(f - f(\eta))^2(\eta) \leq R^E \|f\|^2. \quad (18)$$

This assumption might look artificial but the following lemma shows that Assumption 2 is in fact well behaved.

Lemma 5.1. *Assume Assumption 2. Then, for any $f, g : \Omega \rightarrow \mathbb{R}$ with $\|f\|, \|g\| < \infty$, the corresponding more general estimate holds:*

$$\left| \bar{L}^E[(f - f(\eta))(g - g(\eta))](\eta) \right| \leq R^E \|f\| \|g\|.$$

Proof It is a direct consequence of Lemma 7.3 and Assumption 2.

Now we extend Assumption 2 from the environment to the environment process.

Lemma 5.2. *Under Assumption 2, for any $f : \Omega \rightarrow \mathbb{R}$ with $\|f\| < \infty$,*

$$\bar{L}^{EP}(f - f(\eta))^2(\eta) := \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} S_\epsilon^{EP}(f - f(\eta))^2(\eta) \leq R^{EP} \|f\|^2$$

with $R^{EP} = R^E + \|\alpha\| + 2\|\alpha\|_0$.

Proof As the generator of the environment process is $L^{EP} = L^E + L^S$, with

$$L^S f(\eta) = \sum_{z \in \mathbb{Z}^d} \alpha(\eta, z) [f(\theta_{-z}\eta) - f(\eta)]$$

being the shifts induced by the random walk jumps, all we need to proof is that L^S is a bounded operator in terms of the $\|\cdot\|$ -norm:

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} (L^S f(\eta) - L^S f(\xi)) \\ &= \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \sum_{z \in \mathbb{Z}^d} (\alpha(\eta, z) [f(\theta_{-z}\eta) - f(\eta)] - \alpha(\xi, z) [f(\theta_{-z}\xi) - f(\xi)]) \\ &\leq \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \sum_{z \in \mathbb{Z}^d} (\alpha(\eta, z) - \alpha(\xi, z)) [f(\theta_{-z}\eta) - f(\eta)] \\ &\quad + \sum_{x \in \mathbb{Z}^d} \sup_{(\eta, \xi) \in (\Omega \times \Omega)_x} \sum_{z \in \mathbb{Z}^d} \alpha(\xi, z) [f(\theta_{-z}\eta) - f(\eta) - f(\theta_{-z}\xi) + f(\xi)]. \end{aligned}$$

The first term is easily estimated by $\|\alpha\| \|f\|_{osc} \leq \|\alpha\| \|f\|$. For the second term, it is smaller than

$$\sum_{z \in \mathbb{Z}^d} \sup_{\xi \in \Omega} \alpha(\xi, z) 2 \|f\| =: 2 \|\alpha\|_0 \|f\|.$$

So in total we get

$$\|L^S\|_{\|\cdot\| \rightarrow \|\cdot\|} \leq \|\alpha\| + 2\|\alpha\|_0$$

□

5.2 CLT for the path of the random walk

The CLT for the path of the random walk is proven in two steps. First we show that a martingale which corresponds to fluctuations in a specific direction converges to Brownian motion. Secondly, we use the convergence of those martingales to obtain the full functional central limit theorem.

Proposition 5.3. *Assume Assumption 1b and 2 as well as $\|\alpha\|_2 < \infty$. Fix $v \in \mathbb{R}^d$ with $\|v\| = 1$ and write $f : \Omega \times \mathbb{Z}^d \rightarrow \mathbb{R}$, $f(\eta, x) = \langle x, v \rangle$. Define for each $T > 0$ the martingales*

$$M_T(t) := T^{-\frac{1}{2}} (\mathbb{E} [f(\eta_T, X_T) | \mathfrak{F}_{tT}] - \mathbb{E} [f(\eta_T, X_T) | \mathfrak{F}_0]), \quad 0 \leq t \leq T.$$

Then $(M_T(t))_{0 \leq t \leq 1}$ converges to Brownian motion with variance

$$\int L(\Phi_{0,\infty} - \Phi_{0,\infty}(\eta, 0) + f - f(\eta, 0))^2(\eta, 0) \mu^{EP}(d\eta),$$

where

$$\Phi_{0,\infty} - \Phi_{0,\infty}(\eta, 0) = \sum_{z \in \mathbb{Z}^d} \langle z, v \rangle \int_0^\infty S_t^{EP}[\alpha(\cdot, z)] - S_t^{EP}[\alpha(\cdot, z)](\eta) dt.$$

Proof By Proposition 7.6, the predictable quadratic variation is given by

$$\langle M_T \rangle_t = \frac{1}{T} \int_0^{tT} L(S_{T-s}f - S_{T-s}f(\eta_s, X_s))^2(\eta_s, X_s) ds.$$

We want to take the limit $T \rightarrow \infty$, and to that end we first rewrite the inner term via

$$\begin{aligned} S_t f(\eta, x) &= \int_0^t S_s^{EP} \sum_{z \in \mathbb{Z}^d} [\alpha(\cdot, z) \langle z, v \rangle] (\theta_{-x} \eta) ds + \langle x, v \rangle \\ &= \Phi_{0,t}(\eta, x) + f(\eta, x) \end{aligned}$$

with

$$\Phi_{t,t'}(\eta, x) := \sum_{z \in \mathbb{Z}^d} \langle z, v \rangle \int_t^{t'} S_s^{EP}[\alpha(\cdot, z)](\theta_{-x} \eta) ds$$

for shorter notation. To replace $\Phi_{0,t}$ by $\Phi_{0,\infty}$, we observe that

$$\begin{aligned} &|L[\Phi_{0,t} - \Phi_{0,t}(\eta, x) + f - f(\eta, x)]^2(\eta, x) \\ &\quad - L[\Phi_{0,\infty} - \Phi_{0,\infty}(\eta, x) + f - f(\eta, x)]^2(\eta, x)| \\ &= |L[(\Phi_{t,\infty} - \Phi_{t,\infty}(\eta, x))(\Phi_{0,t} - \Phi_{0,t}(\eta, x) + \Phi_{0,\infty} - \Phi_{0,\infty}(\eta, x) + 2f - 2f(\eta, x))](\eta, x)| \\ &\leq \left(\overline{L}[\Phi_{t,\infty} - \Phi_{t,\infty}(\eta, x)]^2(\eta, x) \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\overline{L}[\Phi_{0,t} - \Phi_{0,t}(\eta, x) + \Phi_{0,\infty} - \Phi_{0,\infty}(\eta, x) + 2f - 2f(\eta, x)]^2(\eta, x) \right)^{\frac{1}{2}}, \end{aligned}$$

where the last line is a consequence of Lemma 7.3. As Φ is effectively a function of the environment process $\theta_{-x}\eta$, we can use Lemma 5.2 for the estimate

$$\bar{L}[\Phi_{t,\infty} - \Phi_{t,\infty}(\eta, x)]^2(\eta, x) \leq R^{EP} \|\Phi_{t,\infty}\|^2.$$

The second factor is estimated similarly after first using $(a+b)^2 \leq 2a^2 + 2b^2$ by

$$\begin{aligned} & 2R^{EP} \|\Phi_{0,t} + \Phi_{0,\infty}\|^2 + 8\bar{L}(f - f(\eta, x))^2(\eta, x) \\ & \leq R^{EP} (\|\Phi_{0,t}\| + \|\Phi_{0,\infty}\|)^2 + 8 \sum_{z \in \mathbb{Z}^d} \alpha(\theta_{-x}\eta, z) \langle z, v \rangle^2. \end{aligned}$$

By the second moment assumption $\|\alpha\|_2 < \infty$ the right summand is bounded, and using Theorem 3.1, $\|\Phi_{0,t}\| \leq C_b \|\alpha\|_1$, and $\|\Phi_{t,\infty}\| \xrightarrow{t \rightarrow \infty} 0$. Coming back to the quadratic variation of the martingale M_T , we can now conclude that

$$\begin{aligned} \lim_{T \rightarrow \infty} \langle M_T \rangle_t &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{tT} L(S_{T-s}f - S_{T-s}f(\eta_s, X_s))^2(\eta_s, X_s) ds \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{tT} L(\Phi_{0,\infty} - \Phi_{0,\infty}(\eta_s, X_s) + f - f(\eta_s, X_s))^2(\eta_s, X_s) ds \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{tT} L(\Phi_{0,\infty} - \Phi_{0,\infty}(\theta_{-X_s}\eta_s, 0) + f - f(\theta_{-X_s}\eta_s, 0))^2(\theta_{-X_s}\eta_s, 0) ds \\ &= t \int L(\Phi_{0,\infty} - \Phi_{0,\infty}(\eta, 0) + f - f(\eta, 0))^2(\eta, 0) \mu^{EP}(d\eta). \end{aligned}$$

Now that we have the converge of the predictable quadratic variation, all that remains to obtain the CLT for the martingale M_T is to show that its jumps vanish in the limit:

$$\lim_{T \rightarrow \infty} \mathbb{E} \sup_{0 \leq t \leq 1} (M_T(t) - M_T(t-))^2 = 0.$$

To obtain this, we use that

$$\begin{aligned} & M_T(t) - M_T(t-) \\ &= T^{-\frac{1}{2}} (S_{T-tT}f(\eta_{tT}, X_{tT}) - S_{T-tT}f(\eta_{tT-}, X_{tT-})) \\ &= T^{-\frac{1}{2}} (\Phi_{0,T-tT}(\eta_{tT}, X_{tT}) - \Phi_{0,T-tT}(\eta_{tT-}, X_{tT-}) + f(\eta_{tT}, X_{tT}) - f(\eta_{tT-}, X_{tT-})) \\ &= T^{-\frac{1}{2}} (\Phi_{0,T-tT}(\eta_{tT}, X_{tT}) - \Phi_{0,T-tT}(\eta_{tT-}, X_{tT-}) + \langle X_{tT} - X_{tT-}, v \rangle). \end{aligned}$$

As we already now that the difference of the two Φ 's is uniformly bounded, it is only

necessary to look at the jumps of X_t :

$$\begin{aligned}
& \frac{1}{T} \mathbb{E} \sup_{0 \leq t \leq T} \|X_t - X_{t-}\|^2 \\
& \leq \frac{\lambda^2}{T} \mathbb{P}(\text{all jumps} \leq \lambda) + \frac{1}{T} \mathbb{E} \sup_{0 \leq t \leq T} \|X_t - X_{t-}\|^2 \mathbb{1}_{\{\text{some jump} > \lambda\}} \\
& \leq \frac{\lambda^2}{T} + \frac{1}{T} \sum_{\|z\| > \lambda} \|z\|^2 \mathbb{P}(\exists 0 \leq t \leq T : X_t - X_{t-} = z) \\
& \leq \frac{\lambda^2}{T} + \frac{1}{T} \sum_{\|z\| > \lambda} \|z\|^2 T \sup_{\eta \in \Omega} \alpha(\eta, z) \\
& = \frac{\lambda^2}{T} + \sum_{\|z\| > \lambda} \sup_{\eta \in \Omega} \alpha(\eta, z) \|z\|^2.
\end{aligned}$$

By the second moment condition $\|\alpha\|_2 < \infty$, the sum on the right converges to 0 as $\lambda \rightarrow \infty$. So when we choose for example $\lambda = T^{\frac{1}{4}}$, the right hand side converges to 0 as $T \rightarrow \infty$, which proves the functional CLT for $(M_T(t))_{0 \leq t \leq 1}$. \square

Theorem 5.4. *Assume Assumption 1b and 2 and $\|\alpha\|_2 < \infty$. Let*

$$v = \int \sum_{z \in \mathbb{Z}^d} \alpha(\eta, z) z \mu^{EP}(d\eta)$$

be the asymptotic speed of the random walk $(X_t)_{t \geq 0}$. Then

$$\left((T^{-\frac{1}{2}}(X_{tT} - vtT))_{0 \leq t \leq 1} \right)$$

converges in probability to a Brownian motion with covariance matrix

$$\Sigma^2 = \int L \left[(A_\eta + id_x) (A_\eta + id_x)^T \right] (\eta, 0) \mu^{EP}(d\eta), \quad (19)$$

where $A_\eta : \Omega \times \mathbb{Z}^d \rightarrow \mathbb{R}^d$ with

$$A_\eta(\xi, x) := \sum_{z \in \mathbb{Z}^d} z \int_0^\infty S_t^{EP}[\alpha(\cdot, z)](\theta_{-x}\xi) - S_t^{EP}[\alpha(\cdot, z)](\eta) dt$$

and $id_x(\xi, x) = x$.

Proof Define the d -dimensional martingale

$$M_T(t) := T^{-\frac{1}{2}} (\mathbb{E} [X_T | \mathfrak{F}_{tT}] - \mathbb{E} [X_T | \mathfrak{F}_0]).$$

By Proposition 5.3 the projection onto any unit vector $u \in \mathbb{R}^d$ converges to a Brownian motion with variance

$$\begin{aligned} & \int L(\langle A_\eta, u \rangle + \langle id_x, u \rangle)^2(\eta, 0) \mu^{EP}(d\eta) \\ &= u^T \int L \left[(A_\eta + id_x)(A_\eta + id_x)^T \right](\eta, 0) \mu^{EP}(d\eta) u. \end{aligned}$$

As the projected martingales are adapted to (\mathfrak{F}_t) , the σ -algebra of M_T , that implies that M_T converges to a Brownian motion with the given covariance matrix. Since $M_T(1) = T^{-\frac{1}{2}}(X_T - \mathbb{E}_{\eta_0,0} X_T)$ we can already conclude a central limit theorem. To obtain the functional central limit theorem, we simply use the fact that $M_T(t)$ is close to $T^{-\frac{1}{2}}(X_{tT} - vtT)$:

$$\begin{aligned} T^{\frac{1}{2}} M_T(t) &= \mathbb{E}[X_T | \eta_{tT}, X_{tT}] - \mathbb{E}[X_T | \eta_0, X_0] \\ &= \int_0^{T-tT} \sum_{z \in \mathbb{Z}^d} z S_s^{EP}[\alpha(\cdot, z)](\theta_{-X_{tT}} \eta_{tT}) + X_{tT} \\ &\quad - \int_0^T \sum_{z \in \mathbb{Z}^d} z S_s^{EP}[\alpha(\cdot, z)](\eta_0). \end{aligned}$$

When we split the last integral at tT , we notice that the part up to tT is close to vtT :

$$\begin{aligned} & \left| \int_0^{tT} \sum_{z \in \mathbb{Z}^d} z S_s^{EP}[\alpha(\cdot, z)](\eta_0) - vtT \right| \\ &= \left| \int_0^{tT} \sum_{z \in \mathbb{Z}^d} z S_s^{EP}[\alpha(\cdot, z)](\eta_0) - \int_0^{tT} \sum_{z \in \mathbb{Z}^d} z S_s^{EP}[\alpha(\cdot, z)](\xi) \mu^{EP}(d\xi) \right| \\ &\leq C_a \|\alpha\|_1 \end{aligned}$$

by Theorem 3.1. Similarly, the part after tT almost annihilates with the integral from 0 to $T - tT$:

$$\begin{aligned} & \left| \int_0^{T-tT} \sum_{z \in \mathbb{Z}^d} z S_s^{EP}[\alpha(\cdot, z)](\theta_{-X_{tT}} \eta_{tT}) - \int_{tT}^T \sum_{z \in \mathbb{Z}^d} z S_s^{EP}[\alpha(\cdot, z)](\eta_0) \right| \\ &= \left| \int_0^{T-tT} \sum_{z \in \mathbb{Z}^d} z S_s^{EP}[\alpha(\cdot, z)](\theta_{-X_{tT}} \eta_{tT}) - \mathbb{E}_{\eta_0,0} \int_0^{T-tT} \sum_{z \in \mathbb{Z}^d} z S_s^{EP}[\alpha(\cdot, z)](\theta_{-X_{tT}} \eta_{tT}) \right| \\ &\leq C_a \|\alpha\|_1. \end{aligned}$$

So

$$\left| M_T(t) - \frac{X_{tT} - vtT}{T^{\frac{1}{2}}} \right| \leq \frac{2C_a \|\alpha\|_1}{T^{\frac{1}{2}}},$$

and the convergence of the drift-adjusted random walk follows to Brownian motion follows from the convergence of M_T . \square

5.3 Some remarks on the variance

A first comment is that the variance (19) is indeed non-degenerate. We remember that the generator L acts as a positive operator, as $A_\eta(\eta, 0) = 0$ and $id_x(\eta, 0) = 0$. By splitting $L = L^E + L^{RW}$, we furthermore see that either $L^E [A_\eta A_\eta^T](\eta)$ is μ^{EP} -a.s. 0, which implies $A_\eta = 0$, or the variance is already positive by contributions from the environment alone. If $A_\eta = 0$, then the second half becomes

$$L^{RW} [id_x id_x^T](\eta, 0) = \sum_{z \in \mathbb{Z}^d} z z^T \alpha(\eta, z),$$

which is 0 if and only if the rates α are 0 μ^{EP} -a.s., in which case the walker is degenerate by construction.

However we can also use the explicit formulation of the variance to study the behaviour as the speed of the environment is increased. For a positive λ denote by

$$L^\lambda = \lambda L^E + L^{RW}$$

the generator where the environment runs at speed λ . By noticing that the new system corresponds to one where the rates are scaled by $1/\lambda$ plus a rescaling of time by λ (or by following the proof of Theorem 3.1) one can see that $A_\eta^\lambda \in O(\lambda^{-1})$. Hence

$$\lim_{\lambda \rightarrow \infty} \lambda L^E [A_\eta^\lambda (A_\eta^\lambda)^T](\eta) = 0$$

and

$$\lim_{\lambda \rightarrow \infty} L^{RW} [(A_\eta^\lambda + id_x)(A_\eta^\lambda + id_x)^T](\eta) = L^{RW} [id_x id_x^T](\eta) = \sum_{z \in \mathbb{Z}^d} z z^T \alpha(\eta, z).$$

As the ergodic measure of the environment process converges to μ^E , the ergodic measure of the environment, we obtain that the variance Σ_λ^2 converges to

$$\int \sum_{z \in \mathbb{Z}^d} z z^T \alpha(\eta, z) \mu^E(d\eta),$$

the variance when averaging of the environment.

6 Concentration estimates

In this section we will obtain detailed results about the deviation from the mean of additive functionals of the environment process and of the position of the walker itself. It is comparatively easy to obtain those from general methods.

Let us do a quick overview of the general ingredients we use here to obtain concentration of additive functionals of Markov processes. Those are general and not specific to this setting. One part is the existence of a suitable estimate of the form

$$[L(g - g(y))^2](y) \leq R \|g\|^2. \quad (20)$$

The second part is an estimate of

$$\left\| \int_0^T S_t f dt \right\| \leq C_T \left\| f \right\|. \quad (21)$$

In general, an estimate of $\left\| \int_0^T S_t f dt \right\|_{osc}$ is also convenient, but in our case that is implied, as $\|g\|_{osc} \leq \left\| g \right\|$ (note that $\left\| \cdot \right\|$ could be any kind of norm-like object in a different context). If estimates (20) and (21) are satisfied, then concentration estimates follow for additive functionals $\int_0^T f(X_t) dt$. The reader is referred to [1] for a more in-depth look at those methods. Section 7 contains similar methods to obtain concentration for $f(X_T)$ instead of additive functionals.

In our context, Assumption 1b guarantees (21) for the environment, and Theorem 3.1,b) lifts that to the environment process. Similarly, Assumption 2 provides (20) for the environment and Lemma 5.2 for the environment process.

6.1 Concentration of additive functionals of the environment process

Theorem 6.1. *Assume Assumptions 1b and 2. Then there exists a constant $D > 0$ so that for any $f : \Omega \rightarrow \mathbb{R}$ with $\left\| f \right\| < \infty$ the deviation estimate*

$$\mathbb{P}_{\nu_1}^{EP} \left(\int_0^T f(\eta_t^{EP}) dt > C_a \left\| f \right\| (r+1) + \int_0^T \nu_2 (S_t^{EP} f) dt \right) \leq e^{-\frac{r^2}{TD}}$$

holds for any $r \geq 0$ and any probability measures ν_1, ν_2 on Ω . In the case that ν_1, ν_2 are equal and point measures, the same holds with $(r+1)$ replaced by r .

The idea of the proof is to use Lemma 7.10. Before starting with the proof, we state a fact which is highly useful to prove the conditions of the general concentration estimates.

Lemma 6.2.

a) Let $f : \Omega \rightarrow \mathbb{R}$ with $\|f\|_{osc} < \infty$. Then

$$\overline{L}[(f - f(\eta))^k](\eta) \leq \|f\|_{osc}^{k-2} \overline{L}(f - f(\eta))^2(\eta);$$

b) Fix $\eta \in \Omega$, and let $g_\eta, f_\eta : E \rightarrow \mathbb{R}$ satisfy $f_\eta(\eta), g_\eta(\eta) = 0$ and $f_\eta \leq g_\eta$. Then

$$\overline{L}f_\eta \leq \overline{L}g_\eta.$$

Proof a) Is a direct consequence of b) with $f_\eta = (f - f(\eta))^k$ and $g_\eta = \|f\|_{osc}^{k-2} (f - f(\eta))^2$.

b) Follows by definition of \overline{L} : $\overline{L}f_\eta = \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} S_\epsilon f_\eta \leq \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} S_\epsilon g_\eta = \overline{L}g_\eta$. \square

Remark This lemma is completely general and not restricted to this context.

Proof of Theorem 6.1 We want to apply Lemma 7.10. So we first prove that

$$\overline{L}^{EP} \left(\int_0^{T'} S_t^{EP} f - S_t^{EP} f(\eta) dt \right)^k (\eta) < D' (C_a \parallel f \parallel)^k \quad (22)$$

for any $T' > 0, \eta \in \Omega$ and $k \in \mathbb{N}, k \geq 2$. Recall that by Theorem 3.1 part a),

$$\left\| \int_0^{T'} S_t^{EP} f \right\|_{osc} \leq C_a \parallel f \parallel.$$

Therewith, using Lemma 6.2 a),

$$\begin{aligned} & \overline{L}^{EP} \left(\int_0^{T'} S_t^{EP} f - S_t^{EP} f(\eta) dt \right)^k (\eta) \\ & \leq (C_a \parallel f \parallel)^{k-2} \overline{L}^{EP} \left(\int_0^{T'} S_t^{EP} f - S_t^{EP} f(\eta) dt \right)^2 (\eta). \end{aligned}$$

By Lemma 5.2,

$$\begin{aligned} & \overline{L}^{EP} \left(\int_0^{T'} S_t^{EP} f - S_t^{EP} f(\eta) dt \right)^2 (\eta) \\ & \leq R^{EP} \parallel \int_0^{T'} S_t^{EP} f - S_t^{EP} f(\eta) dt \parallel^2. \end{aligned}$$

Using Theorem 3.1 part b) for the estimate

$$\parallel \int_0^{T'} S_t^{EP} f - S_t^{EP} f(\eta) dt \parallel \leq C_b \parallel f \parallel,$$

we choose $D' = \frac{R^{EP} C_b^2}{C_a^2}$ to obtain (22). As a direct consequence, Lemma 7.10 implies

$$\mathbb{P}_{\nu_1}^{EP} \left(\int_0^T f(\eta_t^{EP}) dt > C_a \parallel f \parallel (r+1) + \int_0^T \nu_2(S_t^{EP} f) dt \right) \leq e^{-\frac{\frac{1}{2}r^2}{TD' + \frac{1}{3}r}}.$$

Since $\parallel f \parallel \geq \parallel f \parallel_{osc}$, the left-hand-side is in fact 0 for $r \geq \frac{T}{C_a}$. So we can further estimate that probability by replacing $\frac{1}{3}r$ by $\frac{T}{3C_a}$. Choosing $D = 2D' + \frac{2}{3C_a}$ completes the proof. \square

6.2 Concentration estimates for the position of the walker

Note how Theorem 6.1 already gives us a strong concentration property for the rates of the walker X_t . However, it is a bit more tricky to obtain good concentration estimates for the position of the walker itself.

For concentration results for the position of the walker, we need moment conditions on the transition rates of the walker which are comparable to the strength of the concentration estimate. For example, if the jumps are of bounded size or have some exponential moment, then we obtain a concentration estimate which is Gaussian for small and exponential for large deviations. Note that it is impossible to obtain pure Gaussian concentration, as the example of jumps with rate 1 of size 1 to the right, independent of the environment, i.e. a Poisson process, shows.

Theorem 6.3. *Assume Assumptions 1b and 2. Also assume that $\|\alpha\|_1 < \infty$ and that there exists some $M < \infty$ so that the bound $\|\alpha\|_k \leq M^k$ is satisfied for any $k \in \mathbb{N}$. Then there exist constants $c_1, c_2 > 0$ so that for any probability distributions ν_1, ν_2 on Ω and any $r > 0$*

$$\mathbb{P}_{\nu_1,0}(\|X_T - \mathbb{E}_{\nu_2,0} X_T\| > c_1 r + 2dC_a \|\alpha\|_1) \leq 2de^{-\frac{\frac{1}{2}r^2}{Tc_2 + \frac{1}{3}r}}.$$

If only some moments for the jumps of the walker exist, then we still get a concentration estimate, although it is correspondingly weaker.

Theorem 6.4. *Assume Assumptions 1b and 2. Fix $p > 1$. Assume that $\|\alpha\|_1 < \infty$ and $\|\alpha\|_p < \infty$. Then there exist a constant $c > 0$ depending on the process, but not on p , and a constant c_p depending only on p so that for any probability distributions ν_1, ν_2 on Ω and any $r > 0$*

$$\mathbb{P}_{\nu_1,0}(\|X_T - \mathbb{E}_{\nu_2,0} X_T\| > r) \leq c_p \frac{c^p (T^{\frac{p}{2}} + 1)}{r^p}.$$

The remainder of this section will deal with the proofs of the theorems.

The next lemma will verify the conditions of more general theorems in Section 7, from which the results for the walker are immediate.

Lemma 6.5. *Assume Assumptions 1b and 2. For any vector $v \in \mathbb{R}^d$ with $\|v\| = 1$ we consider the function $f_v : \Omega \times \mathbb{Z}^d \rightarrow \mathbb{R}$, $f_v(\eta, x) = \langle x, v \rangle$. Then*

$$\bar{L}(S_t f_v - S_t f_v(\eta, x))^k(\eta, x) \leq 2^k (R^{EP}) C_b^2 C_a^{k-2} \|\alpha\|_1^k + 2^k \|\alpha\|_k^k.$$

Proof We first note that f_v is in the domain of the generator L , with

$$Lf(\eta, x) = \sum_{z \in \mathbb{Z}^d} \alpha(\theta_{-x}\eta, z) \langle z, v \rangle.$$

Since this is uniformly bounded in η and x the condition of Proposition 7.8 is satisfied. Also observe that

$$S_t f_v(\eta, x) = \int_0^t S_r^{EP} \left(\underbrace{\sum_{z \in \mathbb{Z}^d} \alpha(\cdot, z) \langle z, v \rangle}_{=: g(\cdot)} \right) (\theta_{-x}\eta) dr + f_v(\eta, x).$$

With this in mind, using Lemma 6.2

$$\begin{aligned}
& \overline{L}(S_t f_v - S_t f_v(\eta, x))^k(\eta, x) \\
&= \overline{L} \left(\int_0^r S_r^{EP} g(\theta_{-\cdot}) - S_r^{EP} g(\theta_{-x}\eta) dr + f_v - f_v(\eta, x) \right)^k(\eta, x) \\
&\leq 2^k \overline{L} \left(\int_0^r S_r^{EP} g(\theta_{-\cdot}) - S_r^{EP} g(\theta_{-x}\eta) dr \right)^k(\eta, x) + 2^k \overline{L}(f_v - f_v(\eta, x))^k(\eta, x).
\end{aligned}$$

To estimate the right hand term, as f_v is independent of η ,

$$\begin{aligned}
\overline{L}(f_v - f_v(\eta, x))^k(\eta, x) &= \sum_{z \in \mathbb{Z}^d} \alpha(\theta_{-x}\eta, z) < z, v >^k \\
&\leq \sum_{z \in \mathbb{Z}^d} \sup_{\eta \in \Omega} \alpha(\eta, z) \|z\|^k = \|\alpha\|_k^k.
\end{aligned}$$

To treat the left hand term, we first notice that it can be rewritten in terms of the generator of environment process:

$$\overline{L} \left(\int_0^r S_r^{EP} g(\theta_{-\cdot}) - S_r^{EP} g(\theta_{-x}\eta) dr \right)^k(\eta, x) = \overline{L}^{EP} \left(\int_0^r S_r^{EP} g - S_r^{EP} g(\theta_{-x}\eta) dr \right)^k(\theta_{-x}\eta).$$

Next, we use Theorem 3.1,b) with the fact that $\|g\| \leq \|\alpha\|_1$ to obtain that

$$\left\| \int_0^t S_r^{EP} g dr \right\| \leq C_b \|\alpha\|_1.$$

Together with Lemma 5.2 we therefore arrive at

$$\overline{L} \left(\int_0^r S_r^{EP} g(\theta_{-\cdot}) - S_r^{EP} g(\theta_{-x}\eta) dr \right)^2(\eta, x) \leq R^{EP} C_b^2 \|\alpha\|_1^2.$$

Higher moments we reduce to second moments via Lemma 6.2 utilizing Theorem 3.1,a):

$$\begin{aligned}
& \left| \int_0^r S_r^{EP} g(\theta_{-\cdot}) - S_r^{EP} g(\theta_{-x}\eta) dr \right|^k(\eta, x) \\
&\leq C_a^{k-2} \|\alpha\|_1^{k-2} \left(\int_0^r S_r^{EP} g(\theta_{-\cdot}) - S_r^{EP} g(\theta_{-x}\eta) dr \right)^2(\eta, x).
\end{aligned}$$

□

Proof of Theorem 6.3 Let e_i be an orthonormal basis of \mathbb{R}^d , and write $e_{i+d} := -e_i$. Then

$$\begin{aligned}
& \mathbb{P}_{\nu_1,0}(\|X_T - \mathbb{E}_{\nu_2,0} X_T\| > r + 2dC_a \|\alpha\|_1) \\
&\leq \sum_{i=1}^{2d} \mathbb{P}_{\nu_1,0} \left(f_{e_i}(X_T) - \mathbb{E}_{\nu_2,0} f_{e_i}(X_T) > \frac{1}{2d}r + C_a \|\alpha\|_1 \right).
\end{aligned}$$

Those probabilities we can estimate with Corollary 7.9, whose condition is satisfied according to Lemma 6.5 and the conditions of this theorem. At this point we use the fact that

$$S_T f_{e_i}(\eta, 0) - S_T f_{e_i}(\xi, 0) = \int_0^T S_t^{EP} g_i(\eta) - S_t^{EP} g_i(\xi) dt$$

with $g_i(\eta) = \sum_{z \in \mathbb{Z}^d} \alpha(\eta, z) f_{e_i}(z)$ to get the estimate $C_a \|\alpha\|_1$ for the constant a in Corollary 7.9. The constants c_1, c_2 are chosen appropriately. \square

Proof of Theorem 6.4 We will only give a rough sketch of the proof. It is an application of Theorem 7.11 to obtain an estimate on the p th moment plus Markov's inequality. To use the estimate in Theorem 7.11, the general idea is to use the fact that

$$\mathbb{E}_x X_t = \int_0^t S_s^{EP} \left[\sum_{z \in \mathbb{Z}^d} z \alpha(\cdot, z) \right](x) + x,$$

together with applications of Theorem 3.1. The first term on the right hand side of the estimate in Theorem 7.11 is the main contributor and can be estimated by

$$T^{\frac{p}{2}} 2^{\frac{p}{2}} \left[(C_b^p \|\alpha\|_1^p (R^E + \|L^J\|_{\|\cdot\| \rightarrow \|\cdot\|})^p + \|\alpha\|_2^p) \right].$$

The second term is estimated by

$$2^p \left(C_b^p \|\alpha\|_1^p + \|\alpha\|_p^p \right)$$

and the third by

$$(C_a \|\alpha\|_1)^p.$$

Combining various constants, we then get the estimate

$$\mathbb{E}_{\nu_1} \|X_T - \mathbb{E}_{\nu_2} X_T\|^p \leq c_p C^p (T^{\frac{p}{2}} + 1).$$

\square

6.3 Transience and recurrence

Using the concentration estimates and the convergence to Brownian motion, questions like transience or recurrence of the walker can be treated rather easily. The following two statements about transience and recurrence are not meant to be exhaustive, instead they showcase how typical use the stronger statements of the previous sections.

Theorem 6.6 (Transience). *Assume Assumptions 1b and 2 and suppose $\|\alpha\|_{2+\epsilon} < \infty$ for some $\epsilon > 0$. If asymptotic speed $v = \lim_{t \rightarrow \infty} \frac{X_t}{t}$ is non-zero, then the random walk X_t is transient.*

Proof Since $a := \sum_{z \in \mathbb{Z}^d} \sup_{\eta \in \Omega} \alpha(\eta, z) < \infty$, the duration the walker X_t stays at the origin at each visit is at least $\exp(a)$ -distributed. Hence it is sufficient for transience to show that $\mathbb{P}_{0,\eta}(X_t = 0)$ is integrable in t , which can be seen from Theorem 6.4:

$$\mathbb{P}_{\eta,0}(X_t = 0) \leq \mathbb{P}_{\eta,0}(|X_t - vt| \geq vt) \leq \left(c_{2+\epsilon} c^{2+\epsilon} \frac{t^{\frac{2+\epsilon}{2}} + 1}{(vt)^{2+\epsilon}} \right) \wedge 1,$$

where we used the fact that

$$vt = \mathbb{E}_{\mu^{EP},0} X_t.$$

□

Theorem 6.7 (Recurrence). *Let the dimension $d = 1$ and assume Assumptions 1b and 2. Suppose the walker only has jumps of size 1, i.e.*

$$\alpha(\eta, z) = 0 \quad \forall \eta \in E^{b\mathbb{Z}^d}, z \neq \{-1, 1\},$$

and that it has 0 speed. Then X_t is recurrent.

Proof By Theorem 5.4, $T^{-\frac{1}{2}} X_{tT}$ converges to Brownian motion. Hence there exists (with probability 1) an infinite sequence $t_1 < t_2 < \dots$ of times with $X_{t_{2n}} < 0$ and $X_{t_{2n+1}} > 0$, $n \in \mathbb{N}$. As the walker has only jumps of size 1, it will traverse the origin between t_n, t_{n+1} for any $n \in \mathbb{N}$. □

7 Appendix: General concentration results

In this section we will prove general concentration results for Markov processes. Hence we will forget the connection to random walks in dynamic random environments. To avoid a clash of notation, we will assume that F is a polish space, elements of it are denoted x, y , $(Y_t)_{t \geq 0}$ a Feller process on F , with generator A and semigroup $(P_t)_{t \geq 0}$ (on an appropriate space like $\mathcal{C}_0(F), \mathcal{C}_b(F), \mathcal{C}(F), \mathcal{B}(F), \dots$). The canonical filtration is denoted by $(\mathfrak{F}_t)_{t \geq 0}$.

The method of the proofs will use certain martingale approximation.

Notation 7.1. *Given $f : F \rightarrow \mathbb{R}$, we define the martingale*

$$\begin{aligned} M_T^f(t) &:= \mathbb{E} [f(Y_T) | \mathfrak{F}_t] - \mathbb{E} [f(Y_T) | \mathfrak{F}_0], \quad t \in [0, T], \\ &= P_{T-t} f(Y_t) - P_T f(Y_0). \end{aligned}$$

Remark Note that this martingale is different from the canonical martingale $f(Y_t) - \int_0^t A f(Y_s) ds$, but there are some similarities.

To avoid questions regarding the domain of the generator A , we also introduce (non-linear) versions of A in a natural way which are defined for all functions.

Notation 7.2. For any $f : F \rightarrow \mathbb{R}$, define

$$\overline{A}f(x) := \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (P_\epsilon f(x) - f(x));$$

$$\underline{A}f(x) := \liminf_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (P_\epsilon f(x) - f(x)).$$

We call \overline{A} upper generator and \underline{A} lower generator.

Remark

- a) Note that $\overline{A}f(x), \underline{A}f(x) \in \mathbb{R} \cup \{\pm\infty\}$. But if $f \in \text{dom}(A)$, then $\overline{A}f = \underline{A}f = Af$.
- b) If we look at the martingale

$$\widetilde{M}^f(t) = f(Y_t) - \int_0^t Af(Y_s) ds, \quad (23)$$

then its predictable quadratic variation process is given by

$$\begin{aligned} \langle \widetilde{M}^f \rangle_t &= \int_0^t Af^2(Y_s) - 2f(Y_s)Af(Y_s) ds \\ &= \int_0^t [A(f - f(Y_s))^2] (Y_s) ds \end{aligned} \quad (24)$$

under the assumption that $f, f^2 \in \text{dom}(A)$. If that is not the case, then we still have an upper (lower) bound on $\langle \widetilde{M}^f \rangle$ when using the upper generator \overline{A} (lower generator \underline{A}) in (24).

The following lemma shows that $[\overline{A}(f - f(x))(g - g(x))](x)$ still has the property of a quadratic form.

Lemma 7.3. Let, $f, g : F \rightarrow \mathbb{R}$. Then

$$|\overline{A}[(f - f(x))(g - g(x))](x)|^2 \leq \overline{A}(f - f(x))^2(x) \overline{A}(g - g(x))^2(x).$$

Proof of the lemma Utilizing the definition of \overline{A} and Cauchy-Schwarz inequality,

$$\begin{aligned} &|\overline{A}[(f - f(x))(g - g(x))](x)|^2 \\ &\leq \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} |\mathbb{E}_x(f(Y_\epsilon) - f(x))(g(Y_\epsilon) - g(x))|^2 \\ &\leq \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} [\mathbb{E}_x(f(Y_\epsilon) - f(x))^2 \mathbb{E}_x(g(Y_\epsilon) - g(x))^2] \\ &\leq (\overline{A}(f - f(x))^2(x)) (\overline{A}(g - g(x))^2(x)) \end{aligned}$$

□

Remark It is also interesting to note that

$$\langle f, g \rangle_x = A[(f - f(x))(g - g(x))](x)$$

is a positive semi-definite bilinear form. This form can be related to the martingales (23) by their predictable covariation process, which is given by

$$\left\langle \widetilde{M}^f, \widetilde{M}^g \right\rangle_t = \int_0^t \langle f, g \rangle_{Y_s} ds.$$

Before going into details about the martingale M_T^f and its properties we need the following little fact.

Lemma 7.4. Fix $\delta > 0$, $k \in \mathbb{N}, k \geq 2$ and a probability measure μ . Let $f, g : [0, \delta] \rightarrow L^k(\mu)$. Assume that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mu(f(\epsilon)^k) = a \in \mathbb{R}, \quad \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mu(g(\epsilon)^k) = 0.$$

Then

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mu((f(\epsilon) + g(\epsilon))^k) = a.$$

The same holds true for \limsup and \liminf as long as $\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mu(g(\epsilon)^k) = 0$.

Proof First,

$$\mu((f(\epsilon) + g(\epsilon))^k) = \sum_{j=0}^k \binom{k}{j} \mu(f(\epsilon)^j g(\epsilon)^{k-j}).$$

By Hölder's inequality with $p = \frac{k}{j}, q = \frac{k}{k-j}$,

$$\mu(|f(\epsilon)^j g(\epsilon)^{k-j}|) \leq (\mu(f(\epsilon)^k))^{\frac{j}{k}} (\mu(g(\epsilon)^k))^{\frac{k-j}{k}}.$$

Therewith and with the assumptions on the limits,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mu(|f(\epsilon)^j g(\epsilon)^{k-j}|) \leq \left(\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mu(f(\epsilon)^k)^{\frac{j}{k}} \right) \left(\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mu(g(\epsilon)^k)^{\frac{k-j}{k}} \right) = 0$$

for any $j < k$. □

Now we will state and prove the key lemma regarding the martingale 7.1. It is very parallel to Lemma 2.5 in [10], where the identical approach is used to deal with the analogue martingale constructed from $\int_0^T f(Y_t) dt$ instead of $f(Y_T)$.

Lemma 7.5. Fix $k \in \mathbb{N}, k \geq 2$ If

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E}_x (P_{T-t-\epsilon} f(Y_\epsilon) - P_{T-t} f(Y_\epsilon))^k = 0 \quad \forall x \in F,$$

then:

$$a) \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[(M_T^f(t+\epsilon) - M_T^f(t))^k \mid \mathfrak{F}_t \right] = \overline{A} (P_{T-t}f(\cdot) - P_{T-t}f(Y_t))^k (Y_t);$$

$$b) \liminf_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[(M_T^f(t+\epsilon) - M_T^f(t))^k \mid \mathfrak{F}_t \right] = \underline{A} (P_{T-t}f(\cdot) - P_{T-t}f(Y_t))^k (Y_t).$$

Remark If $\sup_{x \in F} \overline{A}f(x) < \infty$, then the condition is satisfied for any $k \geq 2$.

Proof First,

$$\begin{aligned} M_T^f(t+\epsilon) - M_T^f(t) &= P_{T-t-\epsilon}f(Y_{t+\epsilon}) - P_Tf(Y_0) - P_{T-t}f(Y_t) + P_Tf(Y_0) \\ &= P_{T-t-\epsilon}f(Y_{t+\epsilon}) - P_{T-t}f(Y_{t+\epsilon}) + P_{T-t}f(Y_{t+\epsilon}) - P_{T-t}f(Y_t). \end{aligned}$$

By our assumptions and the Markov property,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[(P_{T-t-\epsilon}f(Y_{t+\epsilon}) - P_{T-t}f(Y_{t+\epsilon}))^k \mid \mathfrak{F}_t \right] = 0.$$

Hence, by Lemma 7.4

$$\begin{aligned} \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[(M_T^f(t+\epsilon) - M_T^f(t))^k \mid \mathfrak{F}_t \right] \\ &= \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[(P_{T-t}f(Y_{t+\epsilon}) - P_{T-t}f(Y_t))^k \mid \mathfrak{F}_t \right] \\ &= \overline{A} (P_{T-t}f(\cdot) - P_{T-t}f(Y_t))^k (Y_t). \end{aligned}$$

The argument for \liminf is analogue. \square

Proposition 7.6. Assume that for all $0 < t \leq T$ and all $x \in F$ $(P_t f - P_t f(x))^2 \in \text{dom}(A)$ and

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E}_x (P_{t-\epsilon}f(Y_\epsilon) - P_t f(Y_\epsilon))^2 = 0.$$

Then the predictable quadratic variation of M_T^f is

$$\left\langle M_T^f \right\rangle_t = \int_0^t A (P_{T-s}f(\cdot) - P_{T-s}f(Y_s))^2 (Y_s) ds.$$

Proof By Lemma 7.5,

$$\frac{d}{dt} \left\langle M_T^f \right\rangle_t = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[\left(M_T^f(t+\epsilon) - M_T^f(t) \right)^2 \mid \mathfrak{F}_t \right] = A (P_{T-t}f(\cdot) - P_{T-t}f(Y_t))^2 (Y_t).$$

\square

Theorem 7.7. Assume that for all $k \in \mathbb{N}, k \geq 2$, all $x \in F$ and all $0 < t \leq T$,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E}_x (P_{t-\epsilon}f(Y_\epsilon) - P_t f(Y_\epsilon))^k = 0.$$

Also assume that

$$\sup_{0 \leq t \leq T} \sup_{x \in F} \sum_{k=2}^{\infty} \frac{1}{k!} \bar{A} (P_t f - P_t f(x))^k(x) < \infty.$$

Write

$$\bar{N}_T^f(t) := \exp \left(M_T^f(t) - \int_0^t \sum_{k=2}^{\infty} \frac{1}{k!} \bar{A} (P_{T-s} f - P_{T-s} f(Y_s))^k(Y_s) ds \right), \quad 0 \leq t \leq T;$$

$$\underline{N}_T^f(t) := \exp \left(M_T^f(t) - \int_0^t \sum_{k=2}^{\infty} \frac{1}{k!} \underline{A} (P_{T-s} f - P_{T-s} f(Y_s))^k(Y_s) ds \right), \quad 0 \leq t \leq T.$$

Then $(\bar{N}_T^f(t))_{0 \leq t \leq T}$ is a supermartingale and $(\underline{N}_T^f(t))_{0 \leq t \leq T}$ is a submartingale. If $(P_{T-t} f - P_{T-t} f(x))^k(x) \in \text{dom}(A)$ for all x and k , then both are equal and a martingale.

Proof It is sufficient to prove that $\limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[\bar{N}_T^f(t+\epsilon) - \bar{N}_T^f(t) \mid \mathfrak{F}_t \right] \leq 0$ (or analogue with \liminf for \underline{N}_T^f). For a shorter notation, write

$$\psi(s, x, k) := \bar{A} (P_{T-s} f - P_{T-s} f(x))^k(x).$$

Then

$$\begin{aligned} & \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[\bar{N}_T^f(t+\epsilon) - \bar{N}_T^f(t) \mid \mathfrak{F}_t \right] \\ &= \bar{N}_T^f(t) \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[\exp \left(M_T^f(t+\epsilon) - M_T^f(t) - \int_t^{t+\epsilon} \sum_{k=2}^{\infty} \frac{1}{k!} \psi(s, Y_s, k) ds \right) - 1 \mid \mathfrak{F}_t \right] \\ &\leq \bar{N}_T^f(t) \sum_{l=1}^{\infty} \frac{1}{l!} \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[\left(M_T^f(t+\epsilon) - M_T^f(t) - \int_t^{t+\epsilon} \sum_{k=2}^{\infty} \frac{1}{k!} \psi(s, Y_s, k) ds \right)^l \mid \mathfrak{F}_t \right]. \end{aligned}$$

First, since $\bar{N}_T^f(t) > 0$, we can ignore it. Next, we study the terms individually in l . For $l = 1$, as M_T^f is a martingale, we get just the term

$$- \sum_{k=2}^{\infty} \frac{1}{k!} \psi(t, Y_t, k).$$

For $l > 1$, we use the fact that

$$\int_t^{t+\epsilon} \sum_{k=2}^{\infty} \frac{1}{k!} \psi(s, Y_s, k) ds \leq \epsilon \sup_{0 \leq t \leq T} \sup_{x \in F} \sum_{k=2}^{\infty} \frac{1}{k!} \psi(t, x, k) = \epsilon \cdot \text{const.}$$

Therefore

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[\left(\int_t^{t+\epsilon} \sum_{k=2}^{\infty} \frac{1}{k!} \psi(s, Y_s, k) ds \right)^l \middle| \mathfrak{F}_t \right] = 0,$$

and by Lemma 7.4, we can drop that term. Finally, by Lemma 7.5,

$$\limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E} \left[\left(M_T^f(t+\epsilon) - M_T^f(t) \right)^l \middle| \mathfrak{F}_t \right] = \psi(s, Y_t, l).$$

So the terms for $l \geq 2$ sum to $\sum_{l=2}^{\infty} \frac{1}{l!} \psi(t, Y_t, l)$, which cancels exactly with the term for $l = 1$. \square

As an direct consequence we can estimate the exponential moment of $f(Y_T)$.

Proposition 7.8. *Assume that for all $k \in \mathbb{N}, k \geq 2$, all $x \in F$ and all $0 < t \leq T$,*

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E}_x (P_{t-\epsilon} f(Y_\epsilon) - P_t f(Y_\epsilon))^k = 0.$$

Let ν_1, ν_2 be two arbitrary probability measures on F . Then

$$\mathbb{E}_{\nu_1} e^{f(X_T) - \nu_2(P_T f)} \leq c(\nu_1, \nu_2) \exp \left(\int_0^T \sup_{x \in F} \sum_{k=2}^{\infty} \frac{1}{k!} \bar{A}(P_t f - P_t f(x))^k(x) dt \right),$$

and

$$\mathbb{E}_{\nu_1} e^{f(X_T) - \nu_2(P_T f)} \geq c(\nu_1, \nu_2) \exp \left(\int_0^T \inf_{x \in F} \sum_{k=2}^{\infty} \frac{1}{k!} \underline{A}(P_t f - P_t f(x))^k(x) dt \right),$$

with

$$c(\nu_1, \nu_2) = \int e^{P_T f(x) - \nu_2(P_T f)} \nu_1(dx).$$

Notably, $c(\delta_x, \delta_x) = 1$.

Proof As a first observation,

$$\mathbb{E}_{\nu_1} \left(e^{f(X_T) - \nu_2(P_T f)} \right) = \mathbb{E}_{\nu_1} \left(e^{P_T f(Y_0) - \nu_2(P_T f)} \mathbb{E} \left[e^{M_T^f(T)} \middle| \mathfrak{F}_0 \right] \right).$$

Next, we rewrite the conditional expectation as

$$\mathbb{E} \left[e^{M_T^f(T)} \middle| \mathfrak{F}_0 \right] = \mathbb{E} \left[\bar{N}_T^f(T) e^{\int_0^T \sum_{k=2}^{\infty} \frac{1}{k!} \bar{A}(P_{T-t} f - P_{T-t} f(Y_t))^k(Y_t) dt} \middle| \mathfrak{F}_0 \right],$$

where \overline{N}_T^f is the supermartingale from Theorem 7.7. As \overline{N}_T^f is positive, we can estimate that term by taking the supremum over the possible values of Y_t in the integral, so that we obtain

$$\mathbb{E} \left[e^{M_T^f(T)} \middle| \mathfrak{F}_0 \right] \leq \mathbb{E} \left[\overline{N}_T^f(T) \middle| \mathfrak{F}_0 \right] e^{\int_0^T \sup_{x \in F} \sum_{k=2}^{\infty} \frac{1}{k!} \overline{A}(P_{T-t}f - P_{T-t}f(x))^k(x) dt}.$$

As \overline{N}_T^f is a supermartingale and $\overline{N}_T^f(0) = 1$, we finally arrive at

$$\mathbb{E}_{\nu_1} e^{f(X_T) - \nu_2(P_T f)} \leq c(\nu_1, \nu_2) \exp \left(\int_0^T \sup_{x \in F} \sum_{k=2}^{\infty} \frac{1}{k!} \overline{A}(P_{T-t}f - P_{T-t}f(x))^k(x) dt \right).$$

Reversing the direction of time yields the claim, and the proof for the other bound is analogue. \square

Corollary 7.9. *Let $f : F \rightarrow \mathbb{R}$ be in $\text{dom}(L)$ with $\|Lf\|_{\infty} < \infty$. Assume that*

$$\overline{A}(P_t f - P_t f(x))^k(x) \leq c_1 c_2^k$$

uniformly in $x \in F$ and $t \in [0, T]$. Then, for any $r > 0$,

$$\mathbb{P}_x(f(Y_T) - \mathbb{E}_x f(Y_T) > r) \leq e^{-\frac{\frac{1}{2}(\frac{r}{c_2})^2}{Tc_1 + \frac{1}{3}\frac{r}{c_2}}}.$$

If f is bounded, then for any probability measures ν_1, ν_2 on F ,

$$\mathbb{P}_{\nu_1}(f(Y_T) - \mathbb{E}_{\nu_2} f(Y_T) > r + \|P_T f\|_{osc}) \leq e^{-\frac{\frac{1}{2}(\frac{r}{c_2})^2}{Tc_1 + \frac{1}{3}\frac{r}{c_2}}}.$$

Proof By Markov's inequality,

$$\begin{aligned} \mathbb{P}_{\nu_1}(f(Y_T) - \mathbb{E}_{\nu_2} f(Y_T) > r) &\leq \mathbb{E}_{\nu_1} e^{\lambda f(Y_T) - \mathbb{E}_{\nu_2} \lambda f(Y_T)} e^{-\lambda r} \\ &\leq e^{Tc_1 \sum_{k=2}^{\infty} \lambda^k c_2^k - \lambda r + \lambda \|P_T f\|_{osc}}, \end{aligned}$$

where the last line is the result from Proposition 7.8. Note that $c(\nu_1, \nu_2) \leq e^{\lambda \|P_T f\|_{osc}}$ and $c(\delta_x, \delta_x) = e^0$. For now we continue with the term $\lambda \|P_T f\|_{osc}$ present. Through optimizing λ , the exponent becomes

$$\frac{r - \|P_T f\|_{osc}}{c_2} - \left(Tc_1 + \frac{r - \|P_T f\|_{osc}}{c_2} \right) \log \left(\frac{r - \|P_T f\|_{osc}}{Tc_1 c_2} + 1 \right).$$

By writing $r' = r - \|P_T f\|_{osc}$, we continue with

$$\frac{r'}{c_2} - \left(Tc_1 + \frac{r'}{c_2} \right) \log \left(\frac{r'}{Tc_1 c_2} + 1 \right).$$

This step is not necessary in the case of a fixed initial configuration x , as then $r' = r$. To show that his term is less than $\frac{-\frac{1}{2}(\frac{r'}{c_2})^2}{Tc_1 + \frac{1}{3}\frac{r'}{c_2}}$, we first rewrite it as the following inequality:

$$\log\left(\frac{r'}{Tc_1c_2} + 1\right) \geq \frac{\frac{\frac{1}{2}(\frac{r'}{c_2})^2}{Tc_1 + \frac{1}{3}\frac{r'}{c_2}} + \frac{r'}{c_2}}{Tc_1 + \frac{r'}{c_2}}.$$

Through comparing the derivatives, one concludes that the left hand side is indeed bigger than the right hand side. \square

The next lemma is a slightly refined version of Corollary 2.7 in [10].

Lemma 7.10. *Fix $T > 0$ and $f : Y \rightarrow \mathbb{R}$. Assume that for some $T' > T$ and all $k \in \mathbb{N}, k \geq 2$, $\mathbb{E}_x \sup_{0 \leq t \leq T'} f(Y_t)^k < \infty$. Assume also that for some constants $c_1, c_2 > 0$ the following estimates hold:*

$$\begin{aligned} \sup_{x, y \in X} \sup_{0 \leq T' \leq T} \int_0^{T'} P_t f(x) - P_t f(y) dt &\leq c_1; \\ \sup_{x \in X} \sup_{0 \leq T' \leq T} \overline{A} \left(\int_0^{T'} P_t f - P_t f(x) dt \right)^2 (x) &\leq c_2 c_1^2. \end{aligned}$$

Then

$$\mathbb{P}_{\nu_1} \left(\int_0^T f(X_t) dt > c_1(r+1) + \int_0^T \nu_2(P_t f) dt \right) \leq e^{-\frac{\frac{1}{2}r^2}{Tc_2 + \frac{1}{3}r}}$$

for any $r > 0$ and any probability measures ν_1, ν_2 on F . If ν_1, ν_2 are equal and a Dirac measure, then one can replace $(r+1)$ by r .

The proof of this lemma is analogue to the proof of Corollary 7.9, using the corresponding exponential moment bound in Theorem 2.6 in [10].

Theorem 7.11. *Let $f : F \rightarrow \mathbb{R}$ and fix $T > 0$. Assume that for all $x \in F$ and all $0 < t < T$*

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \mathbb{E}_x (P_{t-\epsilon} f(Y_\epsilon) - P_t f(Y_\epsilon))^2 = 0.$$

Then, for any probability measures ν_1, ν_2 on F and any $p \geq 2$,

$$\begin{aligned} (\mathbb{E}_{\nu_1} |f(Y_T) - \mathbb{E}_{\nu_2} f(Y_T)|^p)^{\frac{1}{p}} &\leq C_p \left[\left(\mathbb{E}_{\nu_1} \left(\int_0^T \overline{A}(P_{T-t} f - P_{T-t} f(X_t))^2 (X_t) dt \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right. \\ &\quad \left. + \left(\mathbb{E}_{\nu_1} \sup_{0 < t \leq T} |P_{T-t} f(Y_t) - P_{T-t} f(Y_{t-})|^p \right)^{\frac{1}{p}} \right] \\ &\quad + \left(\int |P_T f(x) - \nu_2(P_T f)|^p \nu_1(dx) \right)^{\frac{1}{p}}. \end{aligned}$$

The proof is analogue to the proof of Theorem 2.9 in [10].

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